D-Modules of Pure Gaussian Type from the Viewpoint of Enhanced Ind-Sheaves

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Abstract

D-modules of pure Gaussian type are examples of differential systems on the complex projective line with an irregular singularity and as such are subject to the Stokes phenomenon. Recently, the theory of enhanced ind-sheaves and the Riemann–Hilbert correspondence for holonomic D-modules of A. D'Agnolo and M. Kashiwara have stimulated the study of irregular singularities. In this thesis, we apply their result in order to describe D-modules of pure Gaussian type and their Stokes phenomena in a topological way. We use this description for computing – still solely by means of topology – the Fourier–Laplace transform for this class of D-modules. In particular, we recover (with a completely different proof) results of C. Sabbah about the Stokes data attached to the Fourier–Laplace transform, and we show that our methods yield analogous results in more general situations.

Zusammenfassung

D-Moduln vom reinen Gauß-Typ sind Beispiele für differentielle Systeme auf der komplexen projektiven Geraden mit einer irregulären Singularität und unterliegen als solche dem Stokes-Phänomen. Die Theorie der erweiterten Ind-Garben (*enhanced ind-sheaves*) und die Riemann–Hilbert-Korrespondenz für holonome D-Moduln nach A. D'Agnolo und M. Kashiwara haben unlängst das Studium irregulärer Singularitäten neu angeregt. In dieser Arbeit wenden wir deren Resultate an, um D-Moduln vom reinen Gauß-Typ und ihre Stokes-Phänomene auf topologische Weise zu beschreiben. Wir nutzen diese Beschreibung zur – weiterhin rein topologischen – Berechnung der Fourier–Laplace-Transformation für diese Klasse von D-Moduln. Insbesondere erhalten wir (mit einem vollkommen anderen Beweis) Resultate von C. Sabbah über die Stokes-Daten des Fourier–Laplace-transformierten Systems und wir zeigen, dass unsere Methoden entsprechende Resultate in allgemeineren Fällen liefern.

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Introduction

Algebraic Analysis studies systems of linear partial differential equations from an algebraic point of view. The algebraic analogue of such a system is a D-module, a module over some (noncommutative) ring of differential operators. In recent years, new impulses have been given to the study of holonomic D-modules with irregular singularities, relying on a remarkable result by A. D'Agnolo and M. Kashiwara: the Riemann–Hilbert correspondence for holonomic D-modules (see [6]). It generalizes the classical Riemann–Hilbert correspondence for regular holonomic D-modules and allows for a topological investigation of systems with irregular singularities. Various works have been published within the last years, seeking to exploit the power of this new approach in order to reconstruct well-known results in the new framework or treat further cases which have not been solved with the previous methods.

Differential systems of pure Gaussian type have been investigated by C. Sabbah in [38]. They are D-modules on the complex projective line whose only singularity is an irregular one at infinity, where the formal type involves exponents of the form $\frac{c}{2z'^2}$ (for z' a local coordinate at ∞). Therefore, with exponential factors being quadratic monomials (in the local coordinate at 0), they are natural examples of D-modules with irregular singularities. In particular, nontrivial Stokes phenomena may occur at the singularity. They are usually described by linear algebra data (i.e. matrices relating lifts of formal solutions on different sectors).

An interesting but generally difficult problem is the computation of the Fourier–Laplace transform of a D-module. In particular, one aims at describing the Stokes data of the Fourier–Laplace transform of a D-module in terms of the Stokes data of the original system. D-modules of pure Gaussian type are particularly nice examples for studying such a question: They are unramified, and (for example by the stationary phase formula) they are closed under Fourier–Laplace transform, so the Fourier–Laplace transform of such a D-module is again of pure Gaussian type. In his work, C. Sabbah obtained an explicit transformation rule for the Stokes data under the assumption that all the complex parameters c appearing in the exponential factors share the same argument. He defined Stokes data and proved his result in the context of Stokes-filtered local systems. In this thesis, in contrast, we are going to work in the context of enhanced ind-sheaves established by D'Agnolo–Kashiwara.

Before stating the aims of the thesis, we will describe the ingredients and a short history of the topological approach that we are about to apply.

Riemann–Hilbert correspondences and the Stokes phenomenon

In 1900, David Hilbert presented his famous 23 problems at the International Congress of Mathematicians in Paris. The twenty-first of these problems was formulated as follows:

> „Aus der Theorie der linearen Differentialgleichungen mit einer unabhängigen Veränderlichen z möchte ich auf ein wichtiges Problem hinweisen, welches wohl bereits R i e m a n n im Sinne gehabt hat, und welches darin besteht, zu zeigen, daß es stets eine *lineare Differentialgleichung der F u c h s s c h e n Klasse* mit gegebenen singulären Stellen und einer gegebenen Monodromiegruppe giebt.“[1]

It has been found that in this precise formulation the above existence statement is not true. Nevertheless, the idea of this problem – proving some sort of correspondence between differential systems and their solution spaces – has been a starting point for various works in the sequel.

The classical Riemann–Hilbert correspondence, proved independently by M. Kashiwara and Z. Mebkhout (see [18] and [30]), could be considered an answer to a generalized version of Hilbert's orginal question. It states that the holomorphic solution functor defines a (contravariant) equivalence of categories between the bounded derived category of regular holonomic D-modules on a complex manifold X and the bounded derived category of constructible sheaves on X:

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow[\sim]{\mathcal{S}ol_X} \mathrm{D}^{\mathrm{b}}_{\mathrm{constr}}(\mathbb{C}_X).$$

In its non-derived version, it gives an equivalence between the category of regular holonomic D-modules and the category of perverse sheaves. Such an equivalence is a powerful tool since it provides a "dictionary" for translating between problems related to regular holonomic D-modules and problems in topology.

During the last decades a lot of effort has been put into the search for a generalization of this result, where one can drop the regularity condition. A key observation concerning systems with irregular singularities has already been made by G. G. Stokes in 1857 (cf. [40]): In a neigbourhood of such a singularity, formal solutions do not necessarily converge. In general, they admit asymptotic lifts on small sectors around the singularities, but the asymptotic behaviour of holomorphic solutions (even if they are globally defined outside the singularity) may change when one crosses certain angles, the *Stokes directions*. This is nowadays called the Stokes phenomenon, and there are various ways to formally represent

[1] [13], p. 289f.
English translation (from [14], p. 470f.):
"In the theory of linear differential equations with one independent variable z, I wish to indicate an important problem, one which very likely Riemann himself may have had in mind. This problem is as follows: *To show that there always exists a linear differential equation of the Fuchsian class*, with given singular points and monodromic group."

this phenomenon by so-called Stokes data, which describe the relation between the asymptotic lifts of the formal structure on different sectors. In the language of D-modules, this formal structure and its analytic lifts are formalized in the theorems of Levelt–Turrittin and Hukuhara–Turrittin.

It turns out that, defining Stokes data appropriately, they yield a suitable target category for a generalized Riemann–Hilbert correspondence. Such an approach is due to P. Deligne (see [9]), and it provides a Riemann–Hilbert correspondence for (not necessarily regular) holonomic D-modules in dimension 1, with the category of Stokes-filtered local systems as a target category (cf. also [29, 37]).

A different approach was proposed by M. Kashiwara and P. Schapira. They developed the theory of ind-sheaves ([24]), which permits a sheaf-theoretic study of functions with growth conditions, and they showed that this is a first step towards a topological treatment of irregular holonomic D-modules. In 2013, A. D'Agnolo and M. Kashiwara presented their theory of enhanced ind-sheaves (see [6]), combining these methods with the idea of "enhancement" (that is, introducing an additional real variable in the base space) by D. Tamarkin ([41], cf. also [3]). They showed that the category thus defined admits an embedding of the derived category of holonomic D-modules. In other words, there is a fully faithful functor

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \xhookrightarrow{\mathcal{S}ol_X^{\mathrm{E}}} \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X),$$

called the enhanced solution functor and actually extending the holomorphic solution functor from the classical Riemann–Hilbert correspondence. This result is known as the *Riemann–Hilbert correspondence for holonomic D-modules*. The enhanced solution functor $\mathcal{S}ol_X^{\mathrm{E}}$ is not an equivalence, but it is valid in any dimension. Its essential image has been described in work of T. Mochizuki ([32]), although a description in terms of constructibility as in the classical case has (up to now) not been obtained completely.

The construction of the target category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ is technical, but it is related to sheaf theory of vector spaces and hence of a topological nature. The Riemann–Hilbert correspondence for holonomic D-modules can therefore be used to translate problems from D-module theory into topology. Since the functor is fully faithful, the object $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$ associated to a holonomic D-module $\mathcal{M}$ must also contain all the information about the Stokes data attached to $\mathcal{M}$. Even more, in dimension 1 it must be determined by the Stokes data. Therefore, objects in the essential image of $\mathcal{S}ol_X^{\mathrm{E}}$ should be describable in terms of Stokes data. In [6, §9.8], the authors indicate some ideas regarding the representation of enhanced solutions in terms of gluing data. Using these ideas, the theory has since been applied to the study of Stokes phenomena and Fourier–Laplace transforms (see e.g. [26, 7, 4, 17]). Other approaches to the computation of Fourier–Laplace transforms of Stokes data have recently been developed in [31], [33].

Outline of the thesis

Let us now give an overview of the aims and the structure of the thesis. As mentioned above, the Riemann–Hilbert correspondence of A. D'Agnolo and M. Kashiwara is a powerful result and provides new machinery to work with irregular holonomic D-modules. Even in dimension 1, where earlier approaches (like Stokes-filtered local systems) already yielded a Riemann–Hilbert correspondence, the new ansatz can make the situation clearer. Therefore, we employ the theory of enhanced ind-sheaves in the case of D-modules of pure Gaussian type. With regard to the Fourier–Laplace transform, there is another reason for preferring the enhanced solution functor to the approach via Stokes-filtered local systems: The former has better functorialities, and one avoids constructions using sequences of blow-ups when computing direct images (as they were used in [12, 38], for example).

The main goals of this thesis are the following: Eventually, we want to gain an understanding of the Fourier–Laplace transform for D-modules of pure Gaussian type in the framework of enhanced ind-sheaves. In order to carry out the necessary computations, we first need to explicitly describe the enhanced ind-sheaf associated to such a D-module via the Riemann–Hilbert correspondence. Using this description, we can then compute the Fourier–Laplace transform in a purely topological way, computations which in the end reduce to sheaf theory and algebraic topology. We will recover Sabbah's result and show that it can be generalized to a similar statement in a case where we impose less strict conditions on the parameters. Apart from the explicit results we obtain, an important conclusion is the following: The construction of the category of enhanced ind-sheaves is quite technical and it might not be obvious at first sight how to deal with these objects in explicit situations. It is therefore also one of the aims of the thesis to show that this approach yields results in practice.

The results of this thesis have partially been published in the article [15].

In Chapter 1, we recall important notions and results from the theories involved. In particular, we give a very brief summary of the theory of enhanced ind-sheaves and enhanced sheaves. We also prove first statements, which will be of use later. In particular, we describe the structure of enhanced solutions of meromorphic connections on small sectors around an irregular singularity in dimension 1: The isomorphisms from the classical Hukuhara–Turrittin theorem induce decompositions

$$\pi^{-1}\mathbb{C}_S \otimes \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \pi^{-1}\mathbb{C}_S \otimes \bigoplus_{i\in I} \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{E}^{\varphi_i}) \qquad (\blacktriangledown)$$

for a finite family of meromorphic functions $(\varphi_i)_{i\in I}$ and any sufficiently small sector S centred at the singularity. Here $\mathcal{E}^{\varphi_i}$ are exponential D-modules. (The precise statement is given in Proposition 1.12.)

It is the aim of Chapter 2 to describe explicitly the enhanced solutions of a D-module $\mathcal{M}$ of pure Gaussian type. Such a D-module is a meromorphic connection on the Riemann sphere $\mathbb{P}^1$ with only one (irregular) singularity at ∞ and exponential factors of pole order two.

Definition. Let $C \subset \mathbb{C} \smallsetminus \{0\}$ be a finite subset. Then a holonomic $\mathcal{D}_{\mathbb{P}^1}$-module is said to be of *pure Gaussian type C* if

(I) $\mathcal{M}$ is a meromorphic connection with pole at ∞, i.e.

$$\mathcal{M} \simeq \mathcal{M}(*\infty) \quad \text{and} \quad \operatorname{SingSupp}(\mathcal{M}) = \{\infty\},$$

(II) $\mathcal{M}$ has an unramified Levelt–Turrittin decomposition at ∞ of the form

$$\widehat{\mathcal{M}|_\infty} \simeq \bigoplus_{c\in C} \Big(\mathcal{E}^{-\frac{c}{2z'^2}} \otimes^{\mathsf{D}} \mathcal{R}_c\Big)\widehat{\Big|}_\infty \tag{LT}$$

for regular holonomic $\mathcal{D}_{\mathbb{P}^1}$-modules $\mathcal{R}_c$ and a local coordinate z' at ∞.

We develop rigorously the ideas of [6, §9.8] in this case. That is, we investigate the Stokes phenomenon on the topological side of the Riemann–Hilbert correspondence. We will use gluing techniques similar to sheaf theory for the enhanced ind-sheaves involved. More precisely, we model gluing by short exact sequences and distinguished triangles. Although the third object of a distinguished triangle is, in general, only determined uniquely up to a non-unique isomorphism, it will turn out that uniqueness of this isomorphism is always guaranteed in our constructions.

Most of the times, we will work with closed sectors, and a *closed sector at* ∞ is given as

$$S = \{z \in \mathbb{C} \mid R \leq |z| < \infty, \arg z \in [\theta - \varepsilon, \theta + \varepsilon] \text{ if } z \neq 0\} \subseteq \mathbb{C} \subset \mathbb{P}^1$$

with $R \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $\varepsilon \in \mathbb{R}_{\geq 0}$. In particular, although we investigate the singularity at ∞, it will be possible to set up a lot of concepts in the complex plane $\mathbb{C} = \mathbb{P}^1 \smallsetminus \{\infty\}$ with local coordinate $z = \frac{1}{z'}$ at 0. We obtain a description of $\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1}(\mathcal{M})$ in several steps:

▷ **Width of sectors**

First, we prove a result (well-known in the context of Stokes-filtered local systems) about the width of a sector on which a trivialization of the form (▼) exists: If the sector S (centred at ∞) has sufficiently small radius and does not contain more than one Stokes direction for any pair of exponential factors, one can guarantee the existence of a decomposition (see Proposition 2.10)

$$\pi^{-1}\mathbb{C}_S \otimes \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1}(\mathcal{M}) \simeq \bigoplus_{c\in C} (\mathbb{E}_S^{-\operatorname{Re}\frac{c}{2}z^2})^{r_c},$$

where $\mathbb{E}_S^{-\operatorname{Re}\frac{c}{2}z^2} = \pi^{-1}\mathbb{C}_S \otimes \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1}(\mathcal{E}^{-\frac{c}{2}z^2})$ and r_c is the rank of $\mathcal{R}_c$ from (LT). The Stokes directions are determined by the lines where the order (by absolute values) of the functions $e^{-\frac{c}{2}z^2}$ changes, and there are four of them for any pair of exponential factors. In particular, four sectors $S_1, \dots, S_4$ with trivializations

$$\alpha_k \colon \pi^{-1}\mathbb{C}_{S_k} \otimes \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1}(\mathcal{M}) \xrightarrow{\sim} \bigoplus_{c\in C} \big(\mathbb{E}_{S_k}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}.$$

are enough to cover all directions.

▷ **Stokes multipliers**
Once a suitable generic (i.e. non-Stokes) direction θ_0 has been fixed, it determines a numbering of the elements of C if one requires $\operatorname{Re}\frac{c_{(1)}}{2}e^{2\theta_0 i} < \ldots < \operatorname{Re}\frac{c_{(n)}}{2}e^{2\theta_0 i}$. The transition isomorphisms

$$\alpha_{k+1}\circ\alpha_k^{-1} \in \operatorname{Aut}\Big(\bigoplus_{c\in C}\big(\mathbb{E}_{S_k\cap S_{k+1}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}\Big)$$

can be represented by invertible matrices σ_k, which are upper (resp. lower) block-triangular if k is odd (resp. even). These matrices are called *Stokes multipliers* of $\mathcal{M}$ with respect to θ_0. Moreover, it follows that the monodromy is trivial: $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathrm{id}$ (see Proposition 2.13).

▷ **Radius of sectors and reduction to enhanced sheaves**
Since there are no other singularities outside ∞, one can choose the radii of the sectors $S_1, \dots, S_4$ arbitrarily. Hence, we can increase them up to the origin. In this way, the four sectors cover the whole complex plane.

Noting that $\mathbb{E}_S^{-\operatorname{Re}\frac{c}{2}z^2} = \mathbb{C}^{\mathrm{E}}_{\mathbb{P}^1} \overset{+}{\otimes} \mathsf{E}_S^{-\operatorname{Re}\frac{c}{2}z^2}$, where $\mathsf{E}_S^{-\operatorname{Re}\frac{c}{2}z^2} = \mathbb{C}_{\{(z,t)\in\mathbb{C}\times\mathbb{R}\,|\,z\in S,\, t-\operatorname{Re}\frac{c}{2}z^2\geq 0\}} \in \operatorname{Mod}(\mathbb{C}_{\mathbb{C}\times\mathbb{R}})$, one can finally reduce everything to sheaf theory:

Theorem (see Theorem 2.15)**.** *Let $\mathcal{M}$ be of pure Gaussian type C and set $\mathbb{r} = (r_c)_{c\in C}$. Let θ_0 be a generic direction and $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ a family of Stokes multipliers with respect to θ_0. Then*

$$\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1}(\mathcal{M}) \simeq \mathbb{C}^{\mathrm{E}}_{\mathbb{P}^1} \overset{+}{\otimes} \mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}.$$

Here, $\mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}$ is an *enhanced sheaf of pure Gaussian type*, i.e. a sheaf on $\mathbb{C}\times\mathbb{R}$ isomorphic to $\bigoplus_{c\in C}\big(\mathsf{E}_{S_k}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$ on $S_k\times\mathbb{R}$ with transition isomorphisms given by the matrices $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$.

▷ **Stokes data and a Riemann–Hilbert correspondence**
The above theorem is used at the end of the chapter to describe the essential image of D-modules of pure Gaussian type under the functor $\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1}$ and to state and prove a Riemann–Hilbert correspondence for those D-modules (in the spirit of [38]). A suitable category of Stokes data serves as its target category.

Chapter 3 is devoted to the study of the Fourier–Laplace transform. The latter is an integral transform given by

$$^{\mathsf{L}}\mathcal{M} = \mathsf{D}p_{w*}(\mathcal{E}^{-zw} \otimes^{\mathsf{D}} \mathsf{D}p_z^*\mathcal{M})$$

for a $\mathcal{D}_{\mathbb{P}^1_z}$-module $\mathcal{M}$, where $p_z\colon \mathbb{P}^1_z \times \mathbb{P}^1_w \to \mathbb{P}^1_z$ and $p_w\colon \mathbb{P}^1_z \times \mathbb{P}^1_w \to \mathbb{P}^1_w$ denote the projections. (The index denotes the local variable in the chart $\mathbb{C} \subset \mathbb{P}^1$ at 0.)

Let now $\mathcal{M}$ be of pure Gaussian type. Due to the functoriality of the enhanced solution functor and the above theorem, we can conclude

$$\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}^1_w}(^{\mathsf{L}}\mathcal{M}) \simeq \mathbb{C}^{\mathrm{E}}_{\mathbb{P}^1_w} \overset{+}{\otimes} \underbrace{\mathrm{R}\widetilde{q}_{!}\big(\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w} \overset{*}{\otimes} \widetilde{p}^{-1}\mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma}\big)}_{=:\,^{\mathcal{L}}\mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma}}.$$

Here, $\widetilde{p}\colon \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \to \mathbb{C}_z \times \mathbb{R}$ and $\widetilde{q}\colon \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \to \mathbb{C}_w \times \mathbb{R}$ are the projections.

We can therefore determine the Fourier–Laplace transform $^{\mathsf{L}}\mathcal{M}$ of a D-module of pure Gaussian type by computing the topological Fourier–Laplace transform $^{\mathcal{L}}\mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma}$ of the associated enhanced sheaf of pure Gaussian type, and we examine three cases:

1) The simplest D-module of pure Gaussian type, namely the exponential D-module $\mathcal{E}^{-\frac{c}{2}z^2}$ on $\mathbb{P}^1$, will help us to get a first glance at the ingredients of such a computation. In this case, the Stokes phenomenon is trivial and the associated enhanced sheaf is $\mathsf{E}_{\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}$. We will prove that (see Proposition 3.5 and Corollary 3.6)

$$^{\mathcal{L}}\mathsf{E}_{\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2} \simeq \mathsf{E}_{\mathbb{C}}^{\operatorname{Re}\frac{1}{2c}w^2} \qquad \text{and hence} \qquad ^{\mathsf{L}}\mathcal{E}^{-\frac{c}{2}z^2} \simeq \mathcal{E}^{\frac{1}{2c}w^2}.$$

This is done by purely topological methods: The crucial step will be to compute the proper direct image $\mathrm{R}\widetilde{q}_{!}\mathbb{C}_{\{(z,w,t)\in\mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}\,|\,t-\operatorname{Re}(\frac{c}{2}z^2+zw)\geq 0\}}$. Its stalks are the cohomology groups with compact support of the space described by $\check{t} - \operatorname{Re}(\frac{c}{2}z^2 + z\check{w}) \geq 0$ (as a subset of the complex plane $\mathbb{C}_z$, i.e. for fixed $\check{t}$ and $\check{w}$), and these are determined by means of algebraic topology.

2) Next, we consider the case of a parameter set whose elements all have the same argument, which we denote by $\arg C$. Let $C \subset \mathbb{C} \smallsetminus \{0\}$ be a finite subset such that $\arg c = \arg C$ for any $c \in C$. Let $r_c \in \mathbb{Z}_{>0}$ be a positive integer for any $c \in C$ and set $\theta_0 := -\frac{1}{2}\arg C$. Let $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ be a family of four block matrices (with block structure induced by the r_c and θ_0) such that σ_k is upper (resp. lower) block-triangular for k odd (resp. even) and $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathrm{id}$. We prove the following result.

Theorem (see Theorem 3.7)**.** *If we set $\widehat{C} := -1/C$, $\widehat{\theta}_0 := \pi - \theta_0$ and $\widehat{\mathbb{r}} := \big(r_{-\frac{1}{\widehat{c}}}\big)_{\widehat{c}\in\widehat{C}}$, there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{C}\times\mathbb{R}})$*

$$^{\mathcal{L}}\mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma} \simeq \mathcal{F}^{\widehat{C},\widehat{\theta}_0,\widehat{\mathbb{r}}}_{\sigma}.$$

Note in particular that the gluing matrices σ remain the same. As an immediate corollary, we recover a result of C. Sabbah from [38].

Corollary (see Corollary 3.8)**.** *Let $\mathcal{M}$ be of pure Gaussian type C where $\arg c = \arg d$ for any $c, d \in C$. Denote by $\arg C$ the common argument of the elements of C. Then ${}^{\mathsf{L}}\mathcal{M}$ is of pure Gaussian type $\widehat{C} = -1/C$. Moreover, if $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ is a family of Stokes multipliers for $\mathcal{M}$ with respect to the generic direction $\theta_0 = -\frac{1}{2}\arg C$, then σ is also a family of Stokes multipliers for ${}^{\mathsf{L}}\mathcal{M}$ with respect to the generic direction $\widehat{\theta}_0 = \pi - \theta_0$.*

Let us sketch the idea of the proof of the above theorem:

- Firstly, we choose a suitable decomposition of the complex plane $\mathbb{C}_z$ into four sectors $\mathcal{S}_1, \dots, \mathcal{S}_4$. We compute the objects ${}^{\mathcal{L}}\mathsf{E}_{\mathcal{S}_k}^{-\operatorname{Re}\frac{c}{2}z^2}$ and ${}^{\mathcal{L}}\mathsf{E}_{\mathcal{S}_k\cap\mathcal{S}_{k+1}}^{-\operatorname{Re}\frac{c}{2}z^2}$. This is done similarly to the case of a single exponential. However, the spaces whose cohomology groups need to be determined are more complicated here.

- Secondly, writing for short $\mathcal{F} := \mathcal{F}_\sigma^{C,\theta_0,\mathbbm{r}}$, we consider the sequence

  $$0 \longrightarrow \mathcal{F}_{\mathcal{S}_1\cup\mathcal{S}_2} \longrightarrow \mathcal{F}_{\mathcal{S}_1} \oplus \mathcal{F}_{\mathcal{S}_2} \longrightarrow \mathcal{F}_{\mathcal{S}_1\cap\mathcal{S}_2} \longrightarrow 0,$$

  which gives a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{C}\times\mathbb{R}})$

  $${}^{\mathcal{L}}\mathcal{F}_{\mathcal{S}_1\cup\mathcal{S}_2} \longrightarrow {}^{\mathcal{L}}\mathcal{F}_{\mathcal{S}_1} \oplus {}^{\mathcal{L}}\mathcal{F}_{\mathcal{S}_2} \longrightarrow {}^{\mathcal{L}}\mathcal{F}_{\mathcal{S}_1\cap\mathcal{S}_2} \xrightarrow{+1},$$

  whose last two objects have been computed in the first step and which therefore yields a description of the first object.

  Similar distinguished triangles describe ${}^{\mathcal{L}}\mathcal{F}_{\mathcal{S}_3\cup\mathcal{S}_4}$ and finally ${}^{\mathcal{L}}\mathcal{F}$. After choosing a suitable decomposition of the complex plane $\mathbb{C}_w$ into four sectors $\widehat{\mathcal{S}}_1, \dots, \widehat{\mathcal{S}}_4$, one proves that ${}^{\mathcal{L}}\mathcal{F}$ decomposes as $\bigoplus_{c\in C}\big(\mathsf{E}_{\widehat{\mathcal{S}}_k}^{-\operatorname{Re}\frac{1}{2c}w^2}\big)^{r_c}$ with the desired gluing matrices.

3) In the last part, we prove a very similar result in a case with parameters not necessarily on the same half-line through the origin. Precisely, we consider the case of two parameters $c, d \in \mathbb{C}^\times$ where

$$\arg c \in \Big[0, \frac{\pi}{2}\Big) \quad \text{and} \quad \arg(d-c) \in \Big[\arg c, \frac{\pi}{2}\Big). \tag{$\llcorner$}$$

We show that the proof of the case with constant argument can be adapted to this situation. Here, we treat in particular the case of two parameters lying in the first quadrant and ranks $r_c = r_d = 1$ (we therefore suppress the ranks $\mathbbm{r}$ in the notation).

Let $C = \{c, d\} \subset \mathbb{C}^\times$ such that ($\llcorner$) is satisfied. Set $\theta_0 := -\frac{1}{2}\arg c$. Let $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ be a family of four 2×2-matrices such that σ_k is upper (resp. lower) triangular if k is odd (resp. even) and $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathrm{id}$. The result is the following.

Theorem (see Theorem 3.22)**.** *If we set $\widehat{C} := \{-\frac{1}{c}, -\frac{1}{d}\}$ and $\widehat{\theta}_0 := \pi - \theta_0$, there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{C}\times\mathbb{R}})$*

$${}^{\mathcal{L}}\mathcal{F}_\sigma^{C,\theta_0} \simeq \mathcal{F}_\sigma^{\widehat{C},\widehat{\theta}_0}.$$

Corollary (see Corollary 3.23)**.** *Let $\mathcal{M}$ be of pure Gaussian type $C = \{c, d\}$ such that (∟) holds. Then ${}^{\mathsf{L}}\mathcal{M}$ is of pure Gaussian type $\widehat{C} = \{-\frac{1}{c}, -\frac{1}{d}\}$. Moreover, if $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ is a family of Stokes multipliers for $\mathcal{M}$ with respect to the generic direction $\theta_0 = -\frac{1}{2}\arg c$, then σ is also a family of Stokes multipliers for ${}^{\mathsf{L}}\mathcal{M}$ with respect to the generic direction $\widehat{\theta}_0 = \pi - \theta_0$.*

Some of the technical considerations are not given in the main text but can be found in the appendices. Appendix A is devoted to constructible sheaves, in particular constant sheaves on locally closed subsets of a topological space (extended by zero outside these subsets). Appendix B studies cohomology groups with compact support of certain regions appearing in our computations of the topological Fourier–Laplace transforms in Chapter 3. In these appendices, we also recall some notions and well-known properties, and we prove easy consequences which are used in this work or which are helpful for a better understanding of its content.

Chapter 1.

Enhanced ind-sheaves and D-modules

In this preliminary chapter, we clarify notation and recall some notions and concepts from various sources which we will use later. We also state several known results and prove additional facts and corollaries which will be useful for our purposes.

A *good topological space* is a topological space which is Hausdorff, locally compact, second-countable and of finite flabby dimension. In particular, any (topological) manifold is good.

Throughout this thesis, we will denote by $\mathbf{k}$ the field of complex numbers, i.e. $\mathbf{k} = \mathbb{C}$.

1.1. Sheaves and ind-sheaves

Let X be a good topological space.

We denote by $\mathrm{Mod}(\mathbf{k}_X)$ the category of sheaves of $\mathbf{k}$-vector spaces on X and by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ its bounded derived category. Furthermore, for a morphism $f\colon X \to Y$ of good topological spaces, the six Grothendieck operations for sheaves are denoted by $\otimes$, $\mathrm{R}\mathcal{H}om$, $\mathrm{R}f_*$, f^{-1}, $\mathrm{R}f_!$ and $f^!$. (In particular, the non-derived tensor product and inverse image are exact.)

For an embedding $j\colon Z \hookrightarrow X$ of a locally closed subset $Z \subseteq X$, the functor $j_!$ is exact and is called *extension by zero*. For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we set $\mathcal{F}_Z := j_! j^{-1}\mathcal{F}$. If $\mathbf{k}_X$ is the constant sheaf with stalk $\mathbf{k}$, we will write $\mathbf{k}_Z$ instead of $(\mathbf{k}_X)_Z$. With this notation, we have $\mathcal{F}_Z \simeq \mathbf{k}_Z \otimes \mathcal{F}$. (Note that $\mathbf{k}_Z$ may also denote the constant sheaf with stalk $\mathbf{k}$ on Z, and hence we will write $j_!\mathbf{k}_Z$ for $(\mathbf{k}_X)_Z$ if there is risk of confusion.)

We refer to [21] for an exposition of sheaf theory (see also [10]). Throughout the thesis, we will make extensive use of the short exact sequences of sheaves

$$\begin{gathered} 0 \longrightarrow \mathbf{k}_{U_1\cap U_2} \longrightarrow \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \longrightarrow \mathbf{k}_{U_1\cup U_2} \longrightarrow 0, \\ 0 \longrightarrow \mathbf{k}_{Z_1\cup Z_2} \longrightarrow \mathbf{k}_{Z_1} \oplus \mathbf{k}_{Z_2} \longrightarrow \mathbf{k}_{Z_1\cap Z_2} \longrightarrow 0 \end{gathered} \tag{1.1}$$

for open $U_1, U_2 \subseteq X$ and closed $Z_1, Z_2 \subseteq X$ (cf. [21, Proposition 2.3.6]). Essentially, these sequences describe how sheaves defined on two open or closed sets can be glued to a sheaf on the union. In Appendix A, we recall some more details.

In [24], the authors introduced ind-sheaves on X as the category $\mathrm{I}(\mathbf{k}_X)$ of ind-objects in the category $\mathrm{Mod}^c(\mathbf{k}_X)$ of sheaves with compact support (see also [34] for the related notion of subanalytic sheaves). Its bounded derived category is denoted by $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$, and we write $\otimes$, $\mathrm{R}\mathcal{I}hom$, $\mathrm{R}f_*$, f^{-1}, $\mathrm{R}f_{!!}$ and $f^!$ for the six Grothendieck operations for ind-sheaves. There is an exact fully faithful functor $\iota_X\colon \mathrm{Mod}(\mathbf{k}_X) \hookrightarrow \mathrm{I}(\mathbf{k}_X)$. Its derived functor is still fully faithful. Since ι_X does not commute with inductive limits in general, we write "$\varinjlim$" for the inductive limit in $\mathrm{I}(\mathbf{k}_X)$. (For a similar reason, one writes $\mathrm{R}f_{!!}$ instead of $\mathrm{R}f_!$ for the proper direct image for ind-sheaves.)

1.2. Enhanced sheaves and ind-sheaves

The idea of introducing an additional real variable in the base space has been implemented in the context of ind-sheaves in [6] (see also [20] for an analogous construction in the framework of subanalytic sheaves). We sketch the main ingredients in this section.

1.2.1. Enhanced ind-sheaves

We recall the definition of enhanced ind-sheaves on a good topological space X from [6]. The construction is done by two successive localizations. For the theory of localization of triangulated categories, we refer to [25, Chapter 10].

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the two-point compactification of the real line. Denote by $\overline{\pi}\colon X \times \overline{\mathbb{R}} \to X$ the projection. One sets

$$\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\mathbb{R}_\infty}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\overline{\mathbb{R}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\{-\infty,+\infty\}})$$

and defines the category of enhanced ind-sheaves on X as

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\mathbb{R}_\infty})/\{K \mid K \simeq \overline{\pi}^{-1}F \text{ for some } F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)\}.$$

If $f\colon X \to Y$ is a morphism of good topological spaces, one has the six operations for enhanced ind-sheaves $\overset{+}{\otimes}$, $\mathrm{R}\mathcal{I}hom^+$, $\mathrm{E}f_*$, $\mathrm{E}f^{-1}$, $\mathrm{E}f_{!!}$ and $\mathrm{E}f^!$.

Moreover, for $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ we get a well-defined operation on $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$

$$K \mapsto \pi^{-1}L \otimes K, \tag{1.2}$$

denoting by $\pi\colon X \times \mathbb{R} \to X$ the projection. It is induced by the tensor product in the category $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\overline{\mathbb{R}}})$, i.e. we consider $\pi^{-1}L$ extended by zero to $X \times \overline{\mathbb{R}}$ and embedded into the category of ind-sheaves via the functor $\iota_{X\times\overline{\mathbb{R}}}$. By this operation, we can obtain $\pi^{-1}\mathbf{k}_Z \otimes K$ for a locally closed $Z \subseteq X$, and this will be the object to be considered when we want to work with the "restriction of K to Z". (In general, working with the actual restriction via $\mathrm{E}j^{-1}$ along the embedding $j\colon Z \hookrightarrow X$ is not equivalent since the latter does not keep track of the behaviour at the boundary of Z.) In fact, one even has a functor

$$\pi^{-1}(\bullet) \otimes (\bullet)\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X),$$

and it is this functor which allows us to tensor an enhanced ind-sheaf with a short exact sequence of sheaves like (1.1) and hence use gluing techniques similar to those in sheaf theory.

Similarly, there is an operation on $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$ given by

$$K \mapsto \mathrm{R}\mathcal{I}hom(\pi^{-1}L, K)$$

for any $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, and we have an isomorphism (cf. [6, Lemma 3.2.8])

$$\pi^{-1}L \otimes \mathrm{R}\mathcal{I}hom(\pi^{-1}L, K) \simeq \pi^{-1}L \otimes K. \tag{1.3}$$

Moreover, one has compatibility with the convolution product $\overset{+}{\otimes}$ in the following sense: For $K_1, K_2 \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$ and $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one proves that (see [6, Lemma 4.3.1])

$$\pi^{-1}L \otimes (K_1 \overset{+}{\otimes} K_2) \simeq K_1 \overset{+}{\otimes} (\pi^{-1}L \otimes K_2). \tag{1.4}$$

We will tacitly use this compatibility of tensor product and convolution in many places. One also has a projection formula for the tensor product operation (1.2).

Lemma 1.1. *Let $f\colon X \to Y$ be a morphism of good topological spaces and denote by $\pi_X\colon X \times \mathbb{R} \to X$ and $\pi_Y\colon Y \times \mathbb{R} \to Y$ the projections. For $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ and $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$, we have an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$*

$$\pi_Y^{-1}L \otimes \mathrm{E}f_{!!}K \simeq \mathrm{E}f_{!!}(\pi_X^{-1}f^{-1}L \otimes K).$$

(Usually, if it is clear from the context, we will just write π instead of π_X and π_Y later.)

Proof. By [6, Lemma 4.3.2], we have an isomorphism $\pi^{-1}L \otimes K \simeq (\pi^{-1}L \otimes \mathbf{k}_{X\times\{0\}}) \overset{+}{\otimes} K$ (for any good topological space X and any $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$). We can then use the projection formula for the enhanced operations ([6, Proposition 4.5.10]) to obtain

$$\begin{aligned}
\pi_Y^{-1}L \otimes \mathrm{E}f_{!!}K &\simeq (\pi_Y^{-1}L \otimes \mathbf{k}_{Y\times\{0\}}) \overset{+}{\otimes} \mathrm{E}f_{!!}K \\
&\simeq \mathrm{E}f_{!!}\big(\mathrm{E}f^{-1}(\pi_Y^{-1}L \otimes \mathbf{k}_{Y\times\{0\}}) \overset{+}{\otimes} K\big) \\
&\simeq \mathrm{E}f_{!!}\big((\tilde{f}^{-1}\pi_Y^{-1}L \otimes \tilde{f}^{-1}\mathbf{k}_{Y\times\{0\}}) \overset{+}{\otimes} K\big) \\
&\simeq \mathrm{E}f_{!!}\big((\pi_X^{-1}f^{-1}L \otimes \mathbf{k}_{X\times\{0\}}) \overset{+}{\otimes} K\big) \\
&\simeq \mathrm{E}f_{!!}\big(\pi_X^{-1}f^{-1}L \otimes K\big).
\end{aligned}$$

The third isomorphism follows from [21, Proposition 2.3.5] together with the fact that $\mathrm{E}f^{-1}$ is induced by the operation $\tilde{f}^{-1}$ for sheaves (cf. [6, Definition 4.5.6, Remark 3.3.21]), where $\tilde{f} = f \times \mathrm{id}_{\mathbb{R}}$. Finally, one notes that $\pi_Y \circ \tilde{f} = f \circ \pi_X$, and that $\tilde{f}^{-1}\mathbf{k}_{Y\times\{0\}} \simeq \mathbf{k}_{X\times\{0\}}$ (for example by [21, Remark 2.3.11]). □

We set

$$\mathbf{k}_X^{\mathrm{E}} := \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbf{k}_{\{t\geq a\}},$$

where $\{t \geq a\} := \{(x,t) \in X \times \overline{\mathbb{R}} \mid t \in \mathbb{R}, t \geq a\}$. The functor $\mathbf{k}_X^{\mathrm{E}} \overset{+}{\otimes} (\bullet)$ is often called *stabilization functor*.

The duality functor is given by $\mathrm{D}_X^{\mathrm{E}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)^{\mathrm{op}}, K \mapsto \mathrm{R}\mathcal{I}hom^{+}(K, \omega_X^{\mathrm{E}})$, where $\omega_X^{\mathrm{E}} := \pi^{-1}\omega_X \otimes \mathbf{k}_X^{\mathrm{E}}$.

1.2.2. Enhanced sheaves

There is a natural functor $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$ given by $\mathcal{F} \mapsto \mathrm{Q}\iota_{X\times\overline{\mathbb{R}}}j_!\mathcal{F}$, where we denote by $j\colon X\times\mathbb{R} \hookrightarrow X\times\overline{\mathbb{R}}$ the embedding and by Q the quotient functor $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\overline{\mathbb{R}}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$. Objects of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}})$, and in particular objects of $\mathrm{Mod}(\mathbf{k}_{X\times\mathbb{R}})$, will be called *enhanced sheaves on* X, and we will consider enhanced sheaves on X as enhanced ind-sheaves through the above functor without writing it each time.

There is a *convolution product* on $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}})$ defined by

$$\mathcal{F} \overset{*}{\otimes} \mathcal{G} := \mathrm{R}\mu_!\big(q_1^{-1}\mathcal{F} \otimes q_2^{-1}\mathcal{G}\big),$$

where the maps $\mu, q_1, q_2 \colon X \times \mathbb{R}^2 \to X \times \mathbb{R}$ are given by $\mu(x,t_1,t_2) = (x, t_1+t_2)$, $q_1(x,t_1,t_2) = (x,t_1)$ and $q_2(x,t_1,t_2) = (x,t_2)$. Via the natural functor from enhanced sheaves to enhanced ind-sheaves above, it corresponds to the convolution product $\overset{+}{\otimes}$ for enhanced ind-sheaves (see [6, Remark 4.1.3]) and isomorphisms like (1.4) hold accordingly.

Remark. Let us mention that this terminology slightly differs from the one used in [4], for example. There, the category of enhanced sheaves is defined to be the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}})$ consisting of objects $\mathcal{F}$ satisfying $(\mathbf{k}_{\{t\geq 0\}} \oplus \mathbf{k}_{\{t\leq 0\}}) \overset{*}{\otimes} \mathcal{F} \simeq \mathcal{F}$. On this subcategory, the natural functor above is an embedding. Indeed, we actually work in this subcategory since all objects appearing will satisfy this condition.

Notation 1.2. For an enhanced sheaf $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}})$ and a locally closed subset $Z \subseteq X$, we write

$$\mathcal{F}_Z := \mathcal{F}_{Z\times\mathbb{R}} \simeq \pi^{-1}\mathbf{k}_Z \otimes \mathcal{F}.$$

Similarly to the notation for enhanced ind-sheaves, this is in the spirit of thinking about enhanced sheaves as objects on X rather than on $X \times \mathbb{R}$. Moreover, it will simplify and shorten our notation.

1.2.3. Enhanced exponentials

We define exponential enhanced sheaves and ind-sheaves (as introduced in [7]), which will play an important role in our computations later.

Definition 1.3. Let $U \subseteq X$ be an open subset, and let $\varphi, \varphi^-, \varphi^+ \colon U \to \mathbb{R}$ be continuous functions. Let moreover $Z \subseteq U$ be locally closed with $\varphi^-(x) \leq \varphi^+(x)$ for any $x \in Z$. Then we define

$$\mathsf{E}^{\varphi}_{Z|X} := \pi^{-1}\mathbf{k}_Z \otimes \mathbf{k}_{\{t+\varphi\geq 0\}} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}}), \qquad \mathbb{E}^{\varphi}_{Z|X} := \mathbf{k}^{\mathrm{E}}_X \overset{+}{\otimes} \mathsf{E}^{\varphi}_{Z|X} \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X),$$
$$\mathsf{E}^{\varphi^+\rhd\varphi^-}_{Z|X} := \pi^{-1}\mathbf{k}_Z \otimes \mathbf{k}_{\{-\varphi^+\leq t<-\varphi^-\}} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}}), \quad \mathbb{E}^{\varphi^+\rhd\varphi^-}_{Z|X} := \mathbf{k}^{\mathrm{E}}_X \overset{+}{\otimes} \mathsf{E}^{\varphi^+\rhd\varphi^-}_{Z|X} \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X),$$

where we write for short $\{t+\varphi\geq 0\} := \{(x,t)\in X\times\mathbb{R} \mid x\in U, t+\varphi(x)\geq 0\}$, and similarly $\{-\varphi^+\leq t<-\varphi^-\} := \{(x,t)\in X\times\mathbb{R} \mid x\in U, -\varphi^+(x)\leq t<-\varphi^-(x)\}$.

Note that $\mathsf{E}^{\varphi}_{Z|X}$ and $\mathsf{E}^{\varphi^+\rhd\varphi^-}_{Z|X}$ are actually objects in $\mathrm{Mod}(\mathbf{k}_{X\times\mathbb{R}})$, that one has $\mathsf{E}^{\varphi}_{Z|X} \simeq \pi^{-1}\mathbf{k}_Z \otimes \mathsf{E}^{\varphi}_{U|X}$, and that there is a short exact sequence (cf. [21, Proposition 2.3.6])

$$0 \to \mathsf{E}^{\varphi^+\rhd\varphi^-}_{Z|X} \to \mathsf{E}^{\varphi^+}_{Z|X} \to \mathsf{E}^{\varphi^-}_{Z|X} \to 0.$$

The exponential enhanced ind-sheaf $\mathbb{E}^{\varphi}_{Z|X}$ can also be described as (see Lemma A.5)

$$\mathbb{E}^{\varphi}_{Z|X} \simeq \pi^{-1}\mathbf{k}_Z \otimes \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbf{k}_{\{t+\varphi\geq a\}} \simeq \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbf{k}_{\{t+\varphi\geq a\}\cap(Z\times\mathbb{R})}.$$

The following lemma states an important fact about exponential enhanced ind-sheaves, which also follows easily from [7, Corollary 3.2.3] (but we give a more direct proof here). Essentially, it asserts that they are only interesting near a singularity of φ.

Lemma 1.4. *If $U\subseteq X$ is open, $\varphi,\psi\colon U\to\mathbb{R}$ are continuous functions and $Z\subseteq U$ is locally closed such that $\varphi-\psi$ is bounded on Z, then there is a canonical isomorphism*

$$\mathbb{E}^{\varphi}_{Z|X} \simeq \mathbb{E}^{\psi}_{Z|X}.$$

In particular, if φ is bounded on Z, we have

$$\mathbb{E}^{\varphi}_{Z|X} \simeq \pi^{-1}\mathbf{k}_Z \otimes \mathbf{k}^{\mathrm{E}}_X.$$

Proof. Since $\varphi-\psi$ is bounded, we can find $p,q\in\mathbb{R}$ such that $p<\varphi-\psi<q$ on Z. We can assume that $p<0$ and $q>0$. Let us write for short $\{t\geq a-\varphi\}$ instead of $\{t\geq a-\varphi\}\cap(Z\times\mathbb{R})$ etc. in the following.

A canonical morphism from $\mathbb{E}^{\varphi}_{Z|X} \simeq \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbf{k}_{\{t\geq a-\varphi\}} \simeq \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbf{k}_{\{t\geq a+p-\varphi\}}$ to $\mathbb{E}^{\psi}_{Z|X} \simeq \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbf{k}_{\{t\geq a-\psi\}}$ is the one induced by the canonical morphisms $\mathbf{k}_{\{t\geq a+p-\varphi\}} \to \mathbf{k}_{\{t\geq a-\psi\}}$ (cf. Lemma A.2). Note that the canonical morphism thus induced does not depend on the value of p as long as $p<0$ and $p<\varphi-\psi$ on Z.

Similarly, the morphisms $\mathbf{k}_{\{t\geq a-\psi\}} \to \mathbf{k}_{\{t\geq a+q-\varphi\}}$ induce a canonical morphism $\mathbb{E}^{\psi}_{Z|X} \to \mathbb{E}^{\varphi}_{Z|X}$. It is straightforward to check that the composition of these two canonical morphisms is the identity and they are therefore the desired isomorphisms.

The special case follows from the fact that

$$\mathbb{E}^{0}_{Z|X} \simeq \pi^{-1}\mathbf{k}_Z \otimes \left(\mathbf{k}^{\mathrm{E}}_X \overset{+}{\otimes} \mathbf{k}_{\{t\geq 0\}}\right) \simeq \pi^{-1}\mathbf{k}_Z \otimes \mathbf{k}^{\mathrm{E}}_X. \qquad \square$$

1.3. D-modules

In this section, we recall some notions and properties related to analytic D-modules. For details, we refer to [19, 16, 1].

Let X be a complex manifold, $\mathcal{O}_X$ its sheaf of holomorphic functions and $\mathcal{D}_X$ the sheaf of differential operators (meaning linear partial differential operators with holomorphic coefficients) on X.

Denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules and by $\mathrm{D^b}(\mathcal{D}_X)$ its bounded derived category.

For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, one can define its *characteristic variety* $\mathrm{char}(\mathcal{M})$, which is a subset of the cotangent bundle T^*X. Its dimension is at least $\dim X$ and we call a coherent $\mathcal{D}_X$-module *holonomic* if $\dim \mathrm{char}(\mathcal{M}) = \dim X$. Denote by $\mathrm{Mod_{hol}}(\mathcal{D}_X)$ the category of holonomic $\mathcal{D}_X$-modules and by $\mathrm{D^b_{hol}}(\mathcal{D}_X)$ the full subcategory of $\mathrm{D^b}(\mathcal{D}_X)$ whose objects have only holonomic cohomologies. If $p\colon T^*X \to X$ is the natural projection and $T^*_X X \subseteq T^*X$ is the zero section, we define the *singular support* of a coherent $\mathcal{D}_X$-module as $\mathrm{SingSupp}(\mathcal{M}) := p(\mathrm{char}(\mathcal{M}) \smallsetminus T^*_X X)$.

For the notion of regular holonomic D-modules, we refer to [19].

The tensor product (resp. direct and inverse image along a morphism $f\colon X \to Y$ of complex manifolds) for $\mathcal{D}_X$-modules is denoted by $\otimes^{\mathsf{D}}$ (resp. $\mathsf{D}f_*$ and $\mathsf{D}f^*$).

An important class of D-modules are the meromorphic connections. Let $D \subset X$ be a complex hypersurface and denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic functions on X having poles on D at most. For a $\mathcal{D}_X$-module $\mathcal{M}$, we set $\mathcal{M}(*D) := \mathcal{M} \otimes^{\mathsf{D}} \mathcal{O}_X(*D)$.

Definition 1.5. A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called a *meromorphic connection (with poles at D)* if $\mathrm{SingSupp}(\mathcal{M}) = D$ and $\mathcal{M}(*D) \simeq \mathcal{M}$.

1.3.1. Exponential D-modules

Exponential D-modules provide basic examples of meromorphic connections with irregular singularities.

Definition 1.6. Let $D \subset X$ be a closed hypersurface and $\varphi \in \Gamma(X; \mathcal{O}_X(*D))$ a meromorphic function on X with poles on D at most. Set $U := X \smallsetminus D$. Then we define

$$\mathcal{E}^{\varphi}_{U|X} := \big(\mathcal{D}_X/\mathrm{ann}(e^{\varphi})\big)(*D),$$

where $\mathrm{ann}(e^{\varphi})$ is the subsheaf of $\mathcal{D}_X$ whose sections on an open set $V \subseteq X$ are operators P satisfying $Pe^{\varphi} = 0$ on $V \smallsetminus D$.

The module $\mathcal{E}^{\varphi}_{U|X}$ is a meromorphic connection with poles at D. It is regular if and only if φ has no poles.

1.3.2. Formal structure of meromorphic connections in dimension 1

Formal meromorphic connections in one variable are well-understood. A classical result is the theorem of Levelt–Turrittin (see [39], for example), which we will present here in the framework of D-modules (following [5]).

Let X be a complex curve and $\mathcal{M}$ a meromorphic connection with poles at a discrete set of points D. For $p \in X$, we define the formal completion of the stalk at p as

$$\widehat{\mathcal{M}|_p} := \widehat{\mathcal{O}_{X,p}} \otimes_{\mathcal{O}_{X,p}} \mathcal{M}_p,$$

where $\widehat{\mathcal{O}_{X,p}}$ is the completion of $\mathcal{O}_{X,p}$ (hence it is the ring of formal power series in a local coordinate at p). The object $\widehat{\mathcal{M}|_p}$ has the structure of an $(\widehat{\mathcal{O}_{X,p}} \otimes_{\mathcal{O}_{X,p}} \mathcal{D}_{X,p})$-module, where the action of $\widehat{\mathcal{O}_{X,p}}$ is defined by acting on the first factor and the action of $\mathcal{D}_{X,p}$ is given by a product rule.

Definition 1.7. We say that a meromorphic connection $\mathcal{M}$ on a complex curve has a *Levelt–Turrittin decomposition* at a point $p \in X$ if there is an isomorphism of $(\widehat{\mathcal{O}_{X,p}} \otimes_{\mathcal{O}_{X,p}} \mathcal{D}_{X,p})$-modules

$$\widehat{\mathcal{M}|_p} \simeq \bigoplus_{i \in I} \widehat{\left(\mathcal{E}^{\varphi_i}_{U'|X'} \otimes^{\mathsf{D}} \mathcal{R}_i\right)\Big|_p}$$

for an open neighbourhood X' of p, $U' := X' \smallsetminus \{p\}$, a finite index set I, meromorphic functions φ_i on X' having poles at p at most, and regular holonomic $\mathcal{D}_{X'}$-modules $\mathcal{R}_i$. The functions φ_i are called *exponential factors* of $\mathcal{M}$ at p.

By the following theorem, any meromorphic connection on X has (up to a pullback along a ramification map in a neighbourhood of p) a Levelt–Turrittin decomposition at any point $p \in X$.

Theorem 1.8 (Levelt–Turrittin). *Let X be an open disc around 0 in $\mathbb{C}$ and $\mathcal{M}$ a meromorphic connection on X with pole at 0. Then there exists a positive integer $m \in \mathbb{Z}_{>0}$ such that, denoting by $f\colon X \to X, x \mapsto x^m$ the ramification map, the meromorphic connection $\mathsf{D}f^*\mathcal{M}$ has a Levelt–Turrittin decomposition at 0.*

Remark. If $X = \mathbb{P}^1$ is the complex projective line, then the modules $\mathcal{R}_i$ in Definition 1.7 can be chosen to be meromorphic connections (one can just replace $\mathcal{R}_i$ by $\mathcal{R}_i(*p)$ since the exponential is a meromorphic connection). Moreover, they can be chosen to be regular holonomic $\mathcal{D}_X$-modules (i.e. defined on the whole of $\mathbb{P}^1$ rather than on an open disc around p). This follows from the fact that the stalk $(\mathcal{R}_i(*p))_p$ can be extended to a meromorphic connection on $\mathbb{P}^1$ in such a way that outside of p there are only regular singularities (cf. [35, Corollary II.3.7]).

On the other hand, the functions φ_i can be chosen to be principle parts of Laurent series, i.e. local sections of $\mathcal{O}_{\mathbb{P}^1}(*D)/\mathcal{O}_{\mathbb{P}^1}$: If $\varphi_i = \varphi_i^{\mathrm{pos}} + \varphi_i^{\mathrm{neg}}$ (where φ_i^{pos} is the part of the Laurent series containing the non-negative powers of the local variable), we have $\mathcal{E}^{\varphi_i}_{U'|X'} \simeq \mathcal{E}^{\varphi_i^{\mathrm{pos}}}_{U'|X'} \otimes^{\mathsf{D}} \mathcal{E}^{\varphi_i^{\mathrm{neg}}}_{U'|X'} \simeq \mathcal{E}^{\varphi_i^{\mathrm{neg}}}_{U'|X'}$. Even more, φ_i^{neg} is a rational function and hence defines a meromorphic function on the whole of $\mathbb{P}^1$.

1.3.3. Local analytic structure of meromorphic connections in dimension 1

The classical Hukuhara–Turrittin theorem (see [28, Theorem II.D] or [29, Théorème (1.4)], for example) states that the Levelt–Turrittin decomposition can be lifted to an analytic decomposition on small sectors around the singular point. The result is best formulated using blow-up spaces. We follow the presentation in [6].

Let X be a complex curve, $D \subset X$ a discrete set of points, and denote by $\varpi\colon \widetilde{X} \to X$ the (real) blow-up of X at D. Recall (see e.g. [37]) that, locally around a point $p \in D$, the blow-up space $\widetilde{X}$ looks like $S^1 \times [0,1)$ and the blow-up map can be given by $\varpi(e^{i\theta},\rho) = \rho \cdot e^{i\theta}$ (where the right-hand side of this equality describes local coordinates in a chart around p). Hence, one has $\varpi^{-1}(p) = S^1 \times \{0\}$.

Let $\mathcal{A}_{\widetilde{X}}$ be the sheaf of tempered holomorphic functions on $\widetilde{X}$, define $\mathcal{D}^{\mathcal{A}}_{\widetilde{X}} := \mathcal{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{D}_X$, and set $\mathcal{M}^{\mathcal{A}} := \mathcal{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{M}$ for a meromorphic connection $\mathcal{M}$ with poles at D.

Theorem 1.9. *Let $\mathcal{M}$ be a meromorphic connection with poles at D admitting a Levelt–Turrittin decomposition at a point $p \in D$ as in Definition 1.7. Then, for every $x \in \varpi^{-1}(p)$ there exists an open neighbourhood $V \subset \varpi^{-1}(X')$ of x and an isomorphism of $\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}$-modules*

$$\mathcal{M}^{\mathcal{A}} \otimes \mathbf{k}_V \simeq \bigoplus_{i\in I} \Big(\big(\mathcal{E}^{\varphi_i}_{U'|X'}\big)^{\mathcal{A}} \Big)^{r_i} \otimes \mathbf{k}_V,$$

where r_i is the rank[2] *of $\mathcal{R}_i$.*

Note that the isomorphism from Theorem 1.9 may depend on the point x, and the fact that it is in general not possible to extend it to a neighbourhood of the whole circle $\varpi^{-1}(p)$ is known as the *Stokes phenomenon*.

1.4. The Riemann–Hilbert correspondence

The classical Riemann–Hilbert correspondence (see [18]) has been extended to possibly irregular holonomic D-modules in [6]. It can be formulated as follows (where, as always, $\mathbf{k} = \mathbb{C}$).

Theorem 1.10 (cf. [6, Theorem 9.5.3])**.** *Let X be a complex manifold. The functor of enhanced solutions*

$$\mathcal{S}ol^{\mathrm{E}}_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$$

is fully faithful.

Here $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ is the full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$ consisting of $\mathbb{R}$-constructible enhanced ind-sheaves, a notion which we will not explicitly use in this thesis. We refer to [6, Section 4.9] for further details.

In particular, Theorem 1.10 means that the enhanced ind-sheaf $\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M})$ assigned to an object $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ encodes all the information of $\mathcal{M}$.

[2] Outside the singularity p, the regular part of the Levelt–Turrittin decomposition is nonsingular (which follows from [19, Theorem 4.7]) and hence a locally free $\mathcal{O}_X$-module by [19, Proposition 4.43]. (Without loss of generality, p is the only singularity of $\mathcal{R}_i$ in U'.) In other words, $\mathcal{R}_i|_{U'\smallsetminus\{p\}} \simeq \mathcal{O}^{r_i}_{U'\smallsetminus\{p\}}$ for some $r_i \in \mathbb{Z}_{>0}$ and we call r_i the *rank* of $\mathcal{R}_i$.

With the notation introduced in the preceding sections and φ, U as in Definition 1.6, we have (see [6, Corollary 9.4.12])

$$\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{E}_{U|X}^{\varphi}) \simeq \mathbb{E}_{U|X}^{\operatorname{Re}\varphi}. \tag{1.5}$$

Similarly to the classical Riemann–Hilbert correspondence, there is an enhanced De Rham functor

$$\mathcal{DR}_X^{\mathrm{E}} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X),$$

which is connected to the solution functor by duality, i.e. for any $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, we have

$$\mathrm{D}_X^{\mathrm{E}}\big(\mathcal{DR}_X^{\mathrm{E}}(\mathcal{M})\big) \simeq \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})[\dim X].$$

The enhanced solution functor satisfies some functoriality properties with respect to the operations on D-modules and enhanced ind-sheaves.

Proposition 1.11 (see [6, Corollary 9.4.10])**.** *For a morphism $f \colon X \to Y$ of complex manifolds, $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Y)$ and $D \subset X$ a closed hypersurface, we have*

(i) $\mathcal{S}ol_X^{\mathrm{E}}(\mathsf{D}f^*\mathcal{N}) \simeq \mathrm{E}f^{-1}\mathcal{S}ol_Y^{\mathrm{E}}(\mathcal{N})$,

(ii) $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}_1 \otimes^{\mathsf{D}} \mathcal{M}_2) \simeq \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}_1) \overset{+}{\otimes} \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}_2)$,

(iii) $\mathcal{S}ol_X^{\mathrm{E}}\big(\mathcal{M}(*D)\big) \simeq \pi^{-1}\mathbf{k}_{X\smallsetminus D} \otimes \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$.

If in addition $\mathcal{M}$ is quasi-good[3] *and $f|_{\operatorname{supp}\mathcal{M}} \colon \operatorname{supp}\mathcal{M} \to Y$ is proper, we have*

(iv) $\mathcal{S}ol_Y^{\mathrm{E}}(\mathsf{D}f_*\mathcal{M}) \simeq \mathrm{E}f_{!!}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})[d_X - d_Y]$,

where d_X and d_Y are the (complex) dimensions of X and Y, respectively.

Furthermore, due to the compatibility with the holomorphic solution functor (cf. [6, Proposition 9.5.4]), we have $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{O}_X) \simeq \mathbf{k}_X^{\mathrm{E}}$.

Remark. It is worth noting that part (iv) of Proposition 1.11 has originally been stated in the form

$$\mathcal{S}ol_Y^{\mathrm{E}}(\mathsf{D}f_*\mathcal{M})[d_Y] \simeq \mathrm{E}f_*\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})[d_X]$$

in [6, Proposition 9.4.10(ii)]. This is indeed equivalent to the statement above due to the properness assumption on f. To see this, one can use the notion of enhanced support (cf. [6, Notation 4.9.10]): For $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$, one sets

$$\operatorname{supp}^{\mathrm{E}}(K) := \overline{\pi}\big(\operatorname{supp}(\mathrm{R}j_{X!!}\mathrm{L}^{\mathrm{E}}K)\big) \subseteq X.$$

(Here $\mathrm{R}j_{X!!}\mathrm{L}^{\mathrm{E}}$ is the fully faithful left adjoint of the quotient functor $\mathrm{Q} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X\times\overline{\mathbb{R}}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$.) Now one can prove that $\operatorname{supp}^{\mathrm{E}}(\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})) = \operatorname{supp}\mathcal{M}$ in the following way: Let

[3] As stated in [6], a $\mathcal{D}_X$-module $\mathcal{M}$ is said to be *quasi-good* if, for any relatively compact open subset $U \subseteq X$, $\mathcal{M}|_U$ is a sum of coherent $\mathcal{O}_X|_U$-submodules. Various formulations of this definition can be found in [22] and [23], for example.

$U \subseteq X$ be open and denote by $g\colon U \hookrightarrow X$ the embedding. Then $U \cap \operatorname{supp}(\mathcal{M}) = \varnothing$ is equivalent to $\mathrm{D}g^*\mathcal{M} \simeq 0$, which is again equivalent to $\mathrm{E}g^{-1}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq 0$. This in turn is the same as saying $\overline{g}^{-1}\mathrm{R}j_{X!!}\mathrm{L}^{\mathrm{E}}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq 0$ with $\overline{g}\colon U\times\overline{\mathbb{R}} \hookrightarrow X\times\overline{\mathbb{R}}$ the inclusion, which means nothing but $(U\times\overline{\mathbb{R}})\cap\operatorname{supp}(\mathrm{R}j_{X!!}\mathrm{L}^{\mathrm{E}}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})) = \varnothing$, or $U\cap\operatorname{supp}^{\mathrm{E}}(\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})) = \varnothing$.

Hence, $\mathrm{E}f_*$ and $\mathrm{E}f_{!!}$ are interchangeable in the above formula by [6, Proposition 4.9.11].

1.5. Structure of enhanced solutions on small sectors

The local structure of meromorphic connections described in Theorem 1.9 results in a local decomposition of the enhanced ind-sheaf associated to the meromorphic connection via the Riemann–Hilbert correspondence. More precisely, we get a decomposition on a small sector at 0 around any direction. This result, stated in Proposition 1.12 below, is the starting point for investigating the Stokes phenomenon in the framework of enhanced ind-sheaves. In view of the application to D-modules of pure Gaussian type, we only treat the case of unramified meromorphic connections, i.e. the case where $m = 1$ (and hence f is the identity) in Theorem 1.8.

Let $\mathcal{M}$ be a meromorphic connection on a complex curve X with poles at a discrete set $D \subset X$. Since all of the following is local around a singularity, we assume without loss of generality (by considering the situation in a chart around the singular point) that $X \subset \mathbb{C}$ is a small disc around 0 and $D = \{0\}$. As before, the blow-up map of X at D is denoted by $\varpi\colon \widetilde{X} \to X$. We assume in particular that X is small enough such that the exponential factors of $\mathcal{M}$ at 0 are defined and holomorphic on the whole of $X \smallsetminus \{0\}$ (with a pole at 0), so we can choose $X' = X$ in the statements of Section 1.3. Set $U := X \smallsetminus \{0\}$.

Proposition 1.12. *If $\mathcal{M}$ has a Levelt–Turrittin decomposition at 0, then for any direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ there exist constants $\varepsilon, R \in \mathbb{R}_{>0}$, determining an open sector $S_\theta = \{z \in X \mid 0 < |z| < R, \arg z \in (\theta-\varepsilon, \theta+\varepsilon)\}$, such that we have an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$*

$$\pi^{-1}\mathbf{k}_{S_\theta} \otimes \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \pi^{-1}\mathbf{k}_{S_\theta} \otimes \bigoplus_{i\in I} \left(\mathbb{E}_{U|X}^{\operatorname{Re}\varphi_i}\right)^{r_i}.$$

This result is stated without proof in [6, Section 9.8]. It is also given as a corollary of a more general result (namely, in any dimension and taking into account ramification) in [17, Proposition 3.5 and Corollary 3.7]. Nevertheless, we give a similar, direct proof in the unramified case here.

The connection with the decomposition in Theorem 1.9 is made by using a version of the enhanced De Rham functor on the blow-up space (along a normal crossing divisor, which is not a restriction in our one-dimensional situation). As defined in [6], it is given by

$$\mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{N}) := \Omega_{\widetilde{X}}^{\mathrm{E}} \otimes^{\mathsf{L}}_{\mathcal{D}_{\widetilde{X}}^{\mathcal{A}}} \mathcal{N} \tag{1.6}$$

for any $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\widetilde{X}}^{\mathcal{A}})$.

Remark. We will not recall the theory of enhanced ind-sheaves with a ring action here. However, let us make the following remark (for details, see [6, Section 4.10]): The object $\Omega_{\widetilde{X}}^{\mathrm{E}}$

is an enhanced ind-sheaf on $\widetilde{X}$ with a "right action of $\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}$". This actually means that it is an ind-sheaf on $\widetilde{X} \times \overline{\mathbb{R}}$ with a right action of $\beta\varpi^{-1}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}$. The functor $\beta := \beta_{\widetilde{X}\times\overline{\mathbb{R}}} \colon \mathrm{D^b}(\mathbf{k}_{\widetilde{X}\times\overline{\mathbb{R}}}) \to \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\widetilde{X}\times\overline{\mathbb{R}}})$ has been defined in [24] and is suppressed in the notational conventions of [6]. That is, in the notation of [24], the definition of the enhanced De Rham on the blow-up (1.6) reads as

$$\mathcal{DR}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}) := \Omega^{\mathrm{E}}_{\widetilde{X}} \otimes^{\mathsf{L}}_{\beta\varpi^{-1}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}} \beta\varpi^{-1}\mathcal{N}.$$

We have the following lemma, which is the crucial step in proving the proposition.

Lemma 1.13. *Let $V \subseteq \widetilde{X}$ be open and $\mathcal{N} \in \mathrm{D^b}(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}})$. Then there is an isomorphism in $\mathrm{E^b}(\mathrm{I}\mathbf{k}_{\widetilde{X}})$*

$$\pi^{-1}\mathbf{k}_V \otimes \mathcal{DR}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}) \simeq \pi^{-1}\mathbf{k}_V \otimes \mathcal{DR}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N} \otimes \mathbf{k}_{\overline{V}}).$$

Proof. Our calculation takes place in the category $\mathrm{D^b}(\mathrm{I}\mathbf{k}_{\widetilde{X}\times\overline{\mathbb{R}}})$, and we notice that $\pi^{-1}\mathbf{k}_V$ can be replaced by $\varpi^{-1}\mathbf{k}_V$ (since the two are isomorphic in $\mathrm{D^b}(\mathrm{I}\mathbf{k}_{\widetilde{X}\times\mathbb{R}_\infty})$, where the tensor product is still well-defined). Furthermore, $\varpi^{-1}\mathbf{k}_V$ implicitly stands for $\iota_{\widetilde{X}\times\overline{\mathbb{R}}}\varpi^{-1}\mathbf{k}_V = \varpi^{-1}\iota_{\widetilde{X}}\mathbf{k}_V$.

By [24, Lemma 3.3.3] and [24, Proposition 4.2.14], respectively, we have

$$\iota_{\widetilde{X}}\mathbf{k}_V \simeq \text{“}\varinjlim_{U\subset\subset\widetilde{X}}\text{”}\, \mathbf{k}_{V\cap U}$$

and

$$\beta_{\widetilde{X}}\mathbf{k}_{\overline{V}} \simeq \text{“}\varinjlim_{U\subset\subset\widetilde{X}, W\supset\overline{V}}\text{”}\, \mathbf{k}_{U\cap\overline{W}}.$$

Therefore

$$\begin{aligned}\beta_{\widetilde{X}}\mathbf{k}_{\overline{V}} \otimes \iota_{\widetilde{X}}\mathbf{k}_V &\simeq \text{“}\varinjlim_{U\subset\subset\widetilde{X}, U'\subset\subset\widetilde{X}}\text{”}\, \mathbf{k}_{V\cap U\cap U'}\\ &\simeq \iota_{\widetilde{X}}\mathbf{k}_V,\end{aligned}$$

since $U \cap U'$ ranges through the family of all relatively compact open subsets of $\widetilde{X}$ as U and U' do. This now enables us to use [24, Theorem 5.4.19] and obtain (in the notation of [24])

$$\begin{aligned}\pi^{-1}\mathbf{k}_V \otimes \mathcal{DR}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}) &\simeq (\Omega^{\mathrm{E}}_{\widetilde{X}} \otimes^{\mathsf{L}}_{\beta\varpi^{-1}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}} \beta\varpi^{-1}\mathcal{N}) \otimes \iota_{X\times\overline{\mathbb{R}}}\varpi^{-1}\mathbf{k}_V\\ &\simeq (\Omega^{\mathrm{E}}_{\widetilde{X}} \otimes^{\mathsf{L}}_{\beta\varpi^{-1}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}} \beta\varpi^{-1}\mathcal{N}) \otimes (\beta\varpi^{-1}\mathbf{k}_{\overline{V}} \otimes \iota_{X\times\overline{\mathbb{R}}}\varpi^{-1}\mathbf{k}_V)\\ &\simeq \big(\Omega^{\mathrm{E}}_{\widetilde{X}} \otimes^{\mathsf{L}}_{\beta\varpi^{-1}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}} \beta\varpi^{-1}(\mathcal{N} \otimes \mathbf{k}_{\overline{V}})\big) \otimes \iota_{X\times\overline{\mathbb{R}}}\varpi^{-1}\mathbf{k}_V\\ &\simeq \pi^{-1}\mathbf{k}_V \otimes \mathcal{DR}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N} \otimes \mathbf{k}_{\overline{V}}).\end{aligned}$$

□

The enhanced De Rham functor on the blow-up is connected to the enhanced De Rham functor on X by the formula ([6, Corollary 9.2.3])

$$\mathcal{DR}_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathrm{E}\varpi_* \mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{M}^{\mathcal{A}}).$$

Since ϖ is a proper map, we get

$$\mathcal{DR}_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathrm{E}\varpi_{!!} \mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{M}^{\mathcal{A}})$$

by [6, Proposition 4.9.11] (noting that $\mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{M}^{\mathcal{A}})$ is $\mathbb{R}$-constructible, as shown in the proof of [6, Theorem 9.3.2]).

We are now able to give our proof for the decomposition of $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$ on sectors.

Proof of Proposition 1.12. Let $x = (e^{i\theta}, 0) \in \widetilde{X}$, and let $V \subset \widetilde{X}$ be the corresponding open neighbourhood from Theorem 1.9. The preimage in $\widetilde{X}$ of a sector $S_\theta = \{z \in X \mid 0 < |z| < R, \arg z \in (\theta - \varepsilon, \theta + \varepsilon)\}$ at 0 around θ is an open rectangle such that x is the centre of one of the edges (as shown in Fig. 1.1). If we choose ε and R (which determine width and length of the rectangle) sufficiently small, the closure of this square will be a subset of V (since V is open) and therefore by Theorem 1.9

$$\mathcal{M}^{\mathcal{A}} \otimes \mathbf{k}_{\overline{\varpi^{-1}(S_\theta)}} \simeq \bigoplus_{i \in I} \big((\mathcal{E}_{U|X}^{\varphi_i})^{\mathcal{A}}\big)^{r_i} \otimes \mathbf{k}_{\overline{\varpi^{-1}(S_\theta)}}.$$

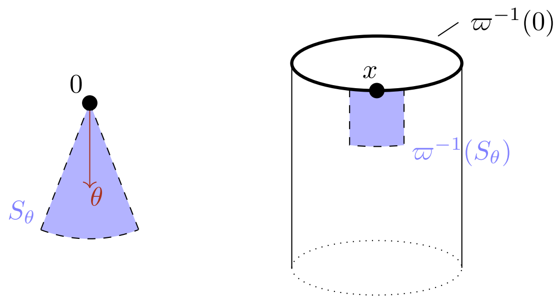


Figure 1.1.: A sector in X and its preimage under the blow-up map $\varpi\colon \widetilde{X} \to X$.

Applying Lemma 1.1 and Lemma 1.13, we get

$$\begin{aligned}
\pi^{-1}\mathbf{k}_{S_\theta} \otimes \mathcal{DR}_X^{\mathrm{E}}(\mathcal{M}) &\simeq \pi^{-1}\mathbf{k}_{S_\theta} \otimes \mathrm{E}\varpi_{!!} \mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{M}^{\mathcal{A}}) \\
&\simeq \mathrm{E}\varpi_{!!}\big(\pi^{-1}\mathbf{k}_{\varpi^{-1}(S_\theta)} \otimes \mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{M}^{\mathcal{A}})\big) \\
&\simeq \mathrm{E}\varpi_{!!}\Big(\pi^{-1}\mathbf{k}_{\varpi^{-1}(S_\theta)} \otimes \mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}\big(\mathcal{M}^{\mathcal{A}} \otimes \mathbf{k}_{\overline{\varpi^{-1}(S_\theta)}}\big)\Big) \\
&\simeq \mathrm{E}\varpi_{!!}\Big(\pi^{-1}\mathbf{k}_{\varpi^{-1}(S_\theta)} \otimes \mathcal{DR}_{\widetilde{X}}^{\mathrm{E}}\Big(\bigoplus_{i \in I} \big((\mathcal{E}_{U|X}^{\varphi_i})^{\mathcal{A}}\big)^{r_i} \otimes \mathbf{k}_{\overline{\varpi^{-1}(S_\theta)}}\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\simeq \mathrm{E}\varpi_{!!}\Big(\pi^{-1}\mathbf{k}_{\varpi^{-1}(S_\theta)} \otimes \mathcal{DR}^{\mathrm{E}}_{\widetilde{X}}\Big(\bigoplus_{i\in I}\big((\mathcal{E}^{\varphi_i}_{U|X})^{\mathcal{A}}\big)^{r_i}\Big)\Big)\\
&\simeq \pi^{-1}\mathbf{k}_{S_\theta} \otimes \bigoplus_{i\in I}\big(\mathcal{DR}^{\mathrm{E}}_X(\mathcal{E}^{\varphi_i}_{U|X})\big)^{r_i}.
\end{aligned}
$$

Dualizing the left hand side of this and using [6, Lemma 4.3.1] yields

$$
\begin{aligned}
\mathrm{D}^{\mathrm{E}}_X\big(\pi^{-1}\mathbf{k}_{S_\theta} \otimes \mathcal{DR}^{\mathrm{E}}_X(\mathcal{M})\big) &\simeq \mathrm{R}\mathcal{I}hom^{+}\big(\pi^{-1}\mathbf{k}_{S_\theta} \otimes \mathcal{DR}^{\mathrm{E}}_X(\mathcal{M}), \omega^{\mathrm{E}}_X\big)\\
&\simeq \mathrm{R}\mathcal{I}hom\big(\pi^{-1}\mathbf{k}_{S_\theta}, \mathrm{R}\mathcal{I}hom^{+}(\mathcal{DR}^{\mathrm{E}}_X(\mathcal{M}), \omega^{\mathrm{E}}_X)\big)\\
&\simeq \mathrm{R}\mathcal{I}hom\big(\pi^{-1}\mathbf{k}_{S_\theta}, \mathrm{D}^{\mathrm{E}}_X\mathcal{DR}^{\mathrm{E}}_X(\mathcal{M})\big)\\
&\simeq \mathrm{R}\mathcal{I}hom\big(\pi^{-1}\mathbf{k}_{S_\theta}, \mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M})\big)[1].
\end{aligned}
$$

An analogous calculation works with the right hand side, and since the shift will be the same, we get

$$
\mathrm{R}\mathcal{I}hom\big(\pi^{-1}\mathbf{k}_{S_\theta}, \mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M})\big) \simeq \mathrm{R}\mathcal{I}hom\Big(\pi^{-1}\mathbf{k}_{S_\theta}, \bigoplus_{i\in I}\big(\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{E}^{\varphi_i}_{U|X})\big)^{r_i}\Big).
$$

The statement of Proposition 1.12 now follows using (1.3) and (1.5). $\square$

Chapter 2.

D-modules of pure Gaussian type

In this chapter, we study differential systems of pure Gaussian type and, in particular, their topological counterpart. These systems were introduced by C. Sabbah in [38], and we give the definition in the language of analytic D-modules on the complex projective line here. We aim at a description of the enhanced ind-sheaf associated to such a D-module via the Riemann–Hilbert correspondence, and such a description is finally achieved in Theorem 2.15. On our way to this result, we introduce standard concepts like Stokes directions and Stokes multipliers in this framework, thus illustrating the study of the Stokes phenomenon from the point of view of enhanced ind-sheaves. A particularly convenient conclusion will be the fact that the description of the enhanced solutions consists in the description of an enhanced sheaf supported on the complex plane, i.e. outside the singularity. As a consequence of our investigation of the Stokes phenomenon, we prove a Riemann–Hilbert correspondence for D-modules of pure Gaussian type, whose target category is a category of linear algebra *Stokes data*, similar to those defined in [38].

2.1. Definition and first properties

Let $\mathbb{P} = \mathbb{P}^1(\mathbb{C})$ be the analytic complex projective line. Denote by $\mathbb{C} = \mathbb{P} \smallsetminus \{\infty\}$ the affine chart with local coordinate z at 0 and by $i\colon \mathbb{C} \hookrightarrow \mathbb{P}$ its embedding.

Definition 2.1 (cf. [38, Definition 1.1])**.** Let $C \subset \mathbb{C}^\times$ be a (nonempty) finite subset. A holonomic $\mathcal{D}_\mathbb{P}$-module $\mathcal{M}$ is said to be of *pure Gaussian type* C if the following conditions hold:

(1) There is an isomorphism of $\mathcal{D}_\mathbb{P}$-modules $\mathcal{M} \simeq \mathcal{M}(*\infty)$.

(2) $\mathrm{SingSupp}(\mathcal{M}) = \{\infty\}$.

(3) There exist regular holonomic $\mathcal{D}_\mathbb{P}$-modules $\mathcal{R}_c$ such that the formal completion of the stalk of $\mathcal{M}$ at ∞ has a Levelt–Turrittin decomposition of the form

$$\widehat{\mathcal{M}|_\infty} \simeq \bigoplus_{c\in C} \left(\mathcal{E}^{-\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{P}} \otimes^{\mathsf{D}} \mathcal{R}_c\right)\widehat{\Big|_\infty}.$$

In other words, $\mathcal{M}$ is a meromorphic connection on $\mathbb{P}$ with pole at ∞ and an (unramified) Levelt–Turrittin decomposition at ∞ with exponential factors of the form $-\frac{c}{2}z^2$. (Note that polynomial functions in z extend to meromorphic functions on $\mathbb{P}$.)

If there exists a finite set C such that $\mathcal{M}$ satisfies (a)–(c), we simply say that $\mathcal{M}$ is of *pure Gaussian type*. The ranks of $\mathcal{R}_c$ will be denoted by r_c, their sum by $r := \sum_{c\in C} r_c$, and the family of these ranks by $\mathbb{r} := (r_c)_{c\in C}$.

Remark. Since i is non-characteristic (see [19, Definition 4.6] for a definition of this notion) for any $\mathcal{D}_{\mathbb{P}}$-module $\mathcal{M}$, condition (2) of Definition 2.1 implies that $\mathrm{SingSupp}(\mathsf{D}i^*\mathcal{M}) = \varnothing$, and hence $\mathsf{D}i^*\mathcal{M} \simeq \mathcal{O}_{\mathbb{C}}^r$ for some $r \in \mathbb{Z}_{>0}$ (by [19, Theorem 4.7, Proposition 4.43]). It follows from Theorem 1.9 that $r = \sum_{c\in C} r_c$ because outside the singularity, $\mathcal{M}$ is locally free, $\mathcal{E}_{\mathbb{C}|\mathbb{P}}^{-\frac{c}{2}z^2}$ is locally free of rank 1, and the rank of a locally free module is unique.

We want to study D-modules of pure Gaussian type and their Stokes phenomena through their enhanced solutions. We already know some properties of the latter.

Lemma 2.2. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be of pure Gaussian type C. Then:*

(i) $\pi^{-1}\mathbf{k}_{\mathbb{C}} \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$.

(ii) Around any direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, there is a small sector S_θ at ∞ such that

$$\pi^{-1}\mathbf{k}_{S_\theta} \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \pi^{-1}\mathbf{k}_{S_\theta} \otimes \bigoplus_{c\in C} \left(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\right)^{r_c}.$$

(iii) For any open $B \subset \mathbb{C}$ such that $\overline{B} \subset \mathbb{C}$ (where $\overline{B}$ denotes the closure of B in $\mathbb{P}$), one has

$$\pi^{-1}\mathbf{k}_B \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \pi^{-1}\mathbf{k}_B \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r.$$

Proof. The statements (i) and (ii) directly follow from Proposition 1.11 and Proposition 1.12, respectively.

The third assertion is proved as follows: It is shown in [8, Lemma 2.7.6] that we have an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})$

$$\pi^{-1}\mathbf{k}_B \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \mathrm{E}j_{B_\infty !!}\mathrm{E}j_{B_\infty}^{-1}\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}).$$

Here $B_\infty = (B, \overline{B})$ is a bordered space, and $j_{B_\infty} \colon (B, \overline{B}) \to \mathbb{P} = (\mathbb{P}, \mathbb{P})$ is the morphism of bordered spaces given by the inclusion. (For details on the theory of bordered spaces, we refer to [6].) If we denote by $j\colon (B, \overline{B}) \to \mathbb{C} = (\mathbb{C}, \mathbb{C})$ the morphism of bordered spaces given by the inclusion (which is well-defined since $\overline{B} \subset \mathbb{C}$) and (as above) by $i\colon \mathbb{C} \to \mathbb{P}$ the embedding, this morphism factors as $j_{B_\infty} = i \circ j$. Hence, we get

$$\begin{aligned}
\pi^{-1}\mathbf{k}_B \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) &\simeq \mathrm{E}j_{B_\infty !!}\mathrm{E}j_{B_\infty}^{-1}\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \\
&\simeq \mathrm{E}j_{B_\infty !!}\mathrm{E}j^{-1}\mathrm{E}i^{-1}\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \\
&\simeq \mathrm{E}j_{B_\infty !!}\mathrm{E}j^{-1}\mathcal{S}ol_{\mathbb{C}}^{\mathrm{E}}(\mathsf{D}i^*\mathcal{M}) \\
&\simeq \mathrm{E}j_{B_\infty !!}\mathrm{E}j^{-1}\mathcal{S}ol_{\mathbb{C}}^{\mathrm{E}}\big(\mathsf{D}i^*(\mathcal{O}_{\mathbb{P}})^r\big) \\
&\simeq \pi^{-1}\mathbf{k}_B \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r.
\end{aligned}$$

□

On the other hand, since $\mathbb{P}$ is compact, we have the following statement about the global structure of $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$. It is a direct application of [4, Lemma 2.5.1], which is again an immediate consequence of $\mathbb{R}$-constructibility of enhanced solutions (cf. [6, Definition 4.9.2, Theorem 4.9.12 and Lemma 9.3.1]).

Lemma 2.3. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be of pure Gaussian type. Denote by $\tilde{i}\colon \mathbb{C}\times\mathbb{R} \hookrightarrow \mathbb{P}\times\mathbb{R}$ the embedding. There exists $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$ such that*

$$\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \tilde{i}_! \mathcal{F}.$$

Thus, the enhanced solutions of a D-module of pure Gaussian type are determined by a globally defined enhanced sheaf which is supported outside the singularity. The aim of the next sections will be to use the properties from Lemma 2.2 in order to describe such an enhanced sheaf.

2.2. Stokes directions and width of sectors

Let $C \subset \mathbb{C}^\times$ be a finite subset and let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be of pure Gaussian type C.

In this section, we extend the decomposition from Lemma 2.2 (ii) to a decomposition of $\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ on sectors around ∞ that intersect at most one Stokes line for each pair $c, d \in C$. That is, we give a more precise description of how "small" the sectors' width has to be – or, how large it may be.

With the following definitions, we introduce the necessary concepts in such a way that everything is set up in $\mathbb{C}$ (with the origin as a center), although the singularity of our module is at ∞. (As we have seen in the preceding section, the enhanced solutions of $\mathcal{M}$ are not interesting at the singularity precisely but in close neighbourhoods, which are then subsets of $\mathbb{C} = \mathbb{P} \smallsetminus \{\infty\}$.)

The Stokes lines are usually defined as the rays from the singularity (i.e. from ∞ in our case) where the order of the absolute values of the functions $e^{-\frac{c}{2}z^2}$ changes. We can also consider them as half-lines emanating from the origin.

Lemma-Definition 2.4. Let $c, d \in C$, $c \neq d$. The set

$$\mathrm{St}_{c,d} := \left\{ z \in \mathbb{C} \,\middle|\, -\operatorname{Re}\frac{c}{2}z^2 = -\operatorname{Re}\frac{d}{2}z^2 \right\}$$

is the union of four closed half-lines with initial point 0, perpendicular to one another. These half-lines are called the *Stokes lines* of the pair c, d. Their directions (i.e. the arguments of the points on the Stokes lines) are called *Stokes directions* (of the pair c, d).

We will say that a set *contains* a direction θ if its intersection with the open half-line $\{z \in \mathbb{C} \smallsetminus \{0\} \mid \arg z = \theta\}$ is not empty. We say that a direction is *generic* if it is not a Stokes direction for any pair of parameters $c, d \in C$.

Proof. Obviously, $0 \in \mathrm{St}_{c,d}$. Furthermore, outside the origin, the condition $-\operatorname{Re}\frac{c}{2}z^2 = -\operatorname{Re}\frac{d}{2}z^2$ is only a restriction on the argument of z since (writing $c = c_1 + ic_2$, $d = d_1 + id_2$)

it is equivalent to

$$(d_1 - c_1)\cos(2\arg z) = (d_2 - c_2)\sin(2\arg z). \tag{2.1}$$

It is easy to see that this equation has exactly four solutions for $\arg z \in \mathbb{R}/2\pi\mathbb{Z}$ and their differences are multiples of $\frac{\pi}{2}$. $\square$

Definition 2.5. A subset $S \subset \mathbb{P}$ is said to be

- an *open sector* at ∞ if

$$S = \left\{ z \in \mathbb{C} \,\middle|\, R < |z| < \infty, \arg z \in (\theta - \varepsilon, \theta + \varepsilon) \right\} \subseteq \mathbb{C} \subset \mathbb{P}$$

  for some $R \in \mathbb{R}_{\geq 0}$, $\varepsilon \in \mathbb{R}_{>0}$, and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.
  (In the chart around ∞ given by $z' = \frac{1}{z}$, this corresponds to an open sector around 0 in the usual sense with radius $\frac{1}{R}$, axis direction $-\theta$ and width 2ε, see Fig. 2.1)

- a *closed sector* at ∞ if

$$S = \left\{ z \in \mathbb{C} \,\middle|\, R \leq |z| < \infty, \arg z \in [\theta - \varepsilon, \theta + \varepsilon] \text{ for } |z| \neq 0 \right\} \subseteq \mathbb{C} \subset \mathbb{P}$$

  for some $R, \varepsilon \in \mathbb{R}_{\geq 0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.
  (For $\varepsilon = 0$, this includes the case of half-lines.)

The *radius* of such a sector is the number $\frac{1}{R} \in (0, +\infty]$, and its *width* is the number $\min(2\varepsilon, 2\pi) \in [0, 2\pi]$. Note that a closed sector at ∞ is topologically closed in $\mathbb{C}$ but not in $\mathbb{P}$, since it does not contain the point ∞.

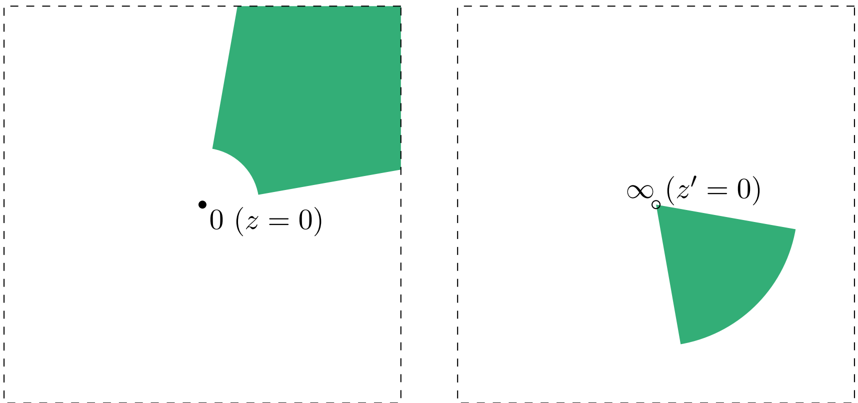


Figure 2.1.: A sector of finite radius at ∞ (as defined in Definition 2.5), drawn in two different charts of $\mathbb{P}$.
On the left: The sector in the chart around 0 with affine coordinate z.
On the right: The same sector in the chart around ∞ with local coordinate z' such that $z' = \frac{1}{z}$. (The vertex is not part of the sector, both in the case of open and closed sectors.)

On sectors containing no Stokes direction, we can introduce an order on C.

Notation 2.6. Let S be a sector at ∞, and let $c, d \in C$. We write

$$c <_S d \quad :\Longleftrightarrow \quad \operatorname{Re} \frac{c}{2} z^2 < \operatorname{Re} \frac{d}{2} z^2 \text{ for all } z \in S \smallsetminus \{0\}.$$

For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we write

$$c <_\theta d \quad :\Longleftrightarrow \quad \operatorname{Re} \frac{c}{2} z^2 < \operatorname{Re} \frac{d}{2} z^2 \text{ for all } z \in \mathbb{C} \text{ with } z \neq 0 \text{ and } \arg z = \theta.$$

We now want to describe morphisms between the exponential enhanced ind-sheaves appearing in the local decomposition of $\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ found in Lemma 2.2 (ii). This will be done in Lemma 2.8 below. For its proof, we need the following lemma from complex analysis about unboundedness of meromorphic functions.

Lemma 2.7. *Let $X \subset \mathbb{C}$ be a connected open subset containing 0 and $\varphi\colon X \to \mathbb{C} \cup \{\infty\}$ a meromorphic function with a pole of strictly positive order at 0. Then $\operatorname{Re}\varphi$ is unbounded on any open sector at 0, i.e. on any $S_{\varepsilon,R}(\theta) = \{z \in \mathbb{C} \smallsetminus \{0\} \mid |z| < R, \arg z \in (\theta - \varepsilon, \theta + \varepsilon)\}$ for arbitrary $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $\varepsilon, R \in \mathbb{R}_{>0}$.*

More precisely, it is unbounded on any open half-line emerging from 0 except for finitely many directions.

Proof. In a sufficiently small punctured neighbourhood of 0, the function φ has a Laurent series expansion and can therefore be written as a product $\varphi(z) = \frac{1}{z^k} g(z)$ where $k \in \mathbb{Z}_{>0}$ is the pole order of φ at zero and g is holomorphic around 0 with $g(0) \neq 0$. In particular, $\operatorname{Re} g$ and $\operatorname{Im} g$ are bounded and we have $\operatorname{Re}\varphi = \operatorname{Re} \frac{1}{z^k} \cdot \operatorname{Re} g - \operatorname{Im} \frac{1}{z^k} \cdot \operatorname{Im} g$.

Now note that $\operatorname{Re} \frac{1}{z^k} = |z|^{-k} \cos(k \cdot \arg z)$ and $\operatorname{Im} \frac{1}{z^k} = -|z|^{-k} \sin(k \cdot \arg z)$. Hence,

$$\operatorname{Re}\varphi(z) = |z|^{-k} \big(\cos(k \cdot \arg z) \cdot \operatorname{Re} g(z) + \sin(k \cdot \arg z) \cdot \operatorname{Im} g(z) \big). \tag{2.2}$$

Now, let z tend to 0 along a half-line, i.e. $|z|$ tends to 0 while $\arg z = \alpha$ remains constant. Then the first factor of (2.2) tends to ∞. The second factor tends to $\cos(k\alpha) \cdot \operatorname{Re} g(0) + \sin(k\alpha) \cdot \operatorname{Im} g(0)$, which is a nonzero (positive or negative) constant except for a finite number of values of α, for which it may be zero. We call these values $\alpha_1, \ldots, \alpha_{2k}$. Thus, on a half-line with $\arg z \notin \{\alpha_1, \ldots, \alpha_{2k}\}$, the function $\operatorname{Re}\varphi$ is unbounded. This concludes the proof since any sector of positive width contains (a part of) such a half-line. □

Remark. Note that in the case where g is constant, i.e. $\varphi(z) = \frac{c}{z^k}$ for some $c \in \mathbb{C}^\times$, the directions in which $\operatorname{Re}\varphi$ is bounded are exactly the half-lines on which $\operatorname{Re}\varphi(z) = 0$. This means that if we apply this lemma to a difference $\varphi(z) = -\frac{c}{2} z^2 + \frac{d}{2} z^2$ of "Gaussian type" meromorphic functions at ∞, the function $\operatorname{Re}\varphi$ is unbounded on any half-line emerging from ∞ except for the Stokes lines.

Lemma 2.8. *Consider the meromorphic functions φ_1, φ_2 on $\mathbb{P}$ given by $\varphi_1(z) = -\frac{c}{2}z^2$ and $\varphi_2(z) = -\frac{d}{2}z^2$ for $c, d \in \mathbb{C}^\times$, $c \neq d$. Let $S \subset \mathbb{P}$ be a sector (open or closed) at ∞. Then we have*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\mathbb{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{P}}, \mathbb{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{P}}\big) &\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\mathsf{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{C}}, \mathsf{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{C}}\big)\\ &\simeq \begin{cases}\mathbf{k} & \text{if } \mathrm{Re}\,\varphi_1 \geq \mathrm{Re}\,\varphi_2 \text{ on } S,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Here, the first isomorphism (from right to left) is induced by the functor $\mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \tilde{i}_!(\bullet)$, and the second isomorphism is the natural identification of a morphism with multiplication by a complex number.

Proof. Using [6, Proposition 4.7.9], we get

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\mathbb{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{P}}, \mathbb{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{P}}\big)\\ &\quad= \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathsf{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{P}}, \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathsf{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{P}}\big)\\ &\quad\simeq \varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\mathsf{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{P}}, \mathbf{k}_{\{t\geq a\}} \overset{+}{\otimes} \mathsf{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{P}}\big)\\ &\quad\simeq \varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}\times\mathbb{R}_\infty})}\big(\mathsf{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{P}}, \mathbf{k}_{\{t\geq a\}} \overset{+}{\otimes} \mathsf{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{P}}\big)\\ &\quad\simeq \varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}\times\overline{\mathbb{R}}})}\big(\mathsf{E}^{\mathrm{Re}\,\varphi_1}_{S|\mathbb{P}}, \mathbf{k}_{\{t\geq a\}} \overset{+}{\otimes} \mathsf{E}^{\mathrm{Re}\,\varphi_2}_{S|\mathbb{P}}\big)\\ &\quad\simeq \varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t+\mathrm{Re}\,\varphi_1\geq 0\}}, \pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t+\mathrm{Re}\,\varphi_2\geq a\}}\big).\end{aligned}$$

The second isomorphism follows from [6, Lemma 4.4.6] together with Lemma A.5 (ii), and the third isomorphism follows from [6, Corollary 3.2.10]. The last isomorphism uses Lemma A.5 (i) as well as the full faithfulness of the embedding from sheaves to ind-sheaves and of extension by zero.

Now, the category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}\times\mathbb{R}})$ can be replaced by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$ since all the sheaves appearing are supported in $\mathbb{C}\times\mathbb{R}$ and extension by zero is fully faithful. Furthermore, note that it suffices to consider $a \geq 0$ in the inductive limit. We claim that this inductive limit is precisely the Hom object with $a = 0$, namely

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t+\mathrm{Re}\,\varphi_1\geq 0\}}, \pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t+\mathrm{Re}\,\varphi_2\geq 0\}}\big).$$

To see this, we distinguish two cases. Set $\Lambda_{j,\geq a} := (S\times\mathbb{R}) \cap \{t \geq a - \mathrm{Re}\,\varphi_j\}$. With this notation, our claim is

$$\varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\mathbf{k}_{\Lambda_{1,\geq 0}}, \mathbf{k}_{\Lambda_{2,\geq a}}\big) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\mathbf{k}_{\Lambda_{1,\geq 0}}, \mathbf{k}_{\Lambda_{2,\geq 0}}\big).$$

- Case 1: $\operatorname{Re}\varphi_1 \geq \operatorname{Re}\varphi_2$ at each point of S.
  Since $a - \operatorname{Re}\varphi_2 \geq -\operatorname{Re}\varphi_1$, we have $\Lambda_{2,\geq a} \subseteq \Lambda_{1,\geq 0}$ for any $a \in \mathbb{R}_{\geq 0}$. Therefore, we can conclude that $\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}(\mathbf{k}_{\Lambda_{1,\geq 0}}, \mathbf{k}_{\Lambda_{2,\geq a}}) \simeq \mathbf{k}$ for any $a \in \mathbb{R}_{\geq 0}$ (cf. Lemma A.2, noting that the $\Lambda_{j,\geq a}$ are closed in $S \times \mathbb{R}$).

- Case 2: In S, there are points where $\operatorname{Re}\varphi_1 < \operatorname{Re}\varphi_2$.
  It follows from Lemma 2.7 (and the subsequent remark) that $\operatorname{Re}(\varphi_2 - \varphi_1)$ is not bounded from above on S. For any $a \in \mathbb{R}_{\geq 0}$, we can thus choose a point $x \in S$ such that $\operatorname{Re}(\varphi_2(x) - \varphi_1(x)) > a$ and set $t := a - \operatorname{Re}\varphi_2(x)$. Then $(x,t) \in \Lambda_{2,\geq a}$ but $(x,t) \notin \Lambda_{1,\geq 0}$, so $\Lambda_{2,\geq a} \nsubseteq \Lambda_{1,\geq 0}$ for any $a \in \mathbb{R}_{\geq 0}$. Therefore, $\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}(\mathbf{k}_{\Lambda_{1,\geq 0}}, \mathbf{k}_{\Lambda_{2,\geq a}}) \simeq 0$ for any $a \in \mathbb{R}_{\geq 0}$.

It remains to show that the isomorphism

$$\operatorname{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\mathbb{E}^{\operatorname{Re}\varphi_1}_{S|\mathbb{P}}, \mathbb{E}^{\operatorname{Re}\varphi_2}_{S|\mathbb{P}}\big) \simeq \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}\times\mathbb{R}})}\big(\mathsf{E}^{\operatorname{Re}\varphi_1}_{S|\mathbb{P}}, \mathsf{E}^{\operatorname{Re}\varphi_2}_{S|\mathbb{P}}\big).$$

is induced by $\mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} (\bullet)$ in the first case. Since $\mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} (\bullet)$ induces a $\mathbf{k}$-linear map between the one-dimensional Hom spaces, it suffices to note that the canonical morphism $\mathsf{E}^{\operatorname{Re}\varphi_1}_{S|\mathbb{P}} \to \mathsf{E}^{\operatorname{Re}\varphi_2}_{S|\mathbb{P}}$ is mapped to a nonzero morphism $\mathbb{E}^{\operatorname{Re}\varphi_1}_{S|\mathbb{P}} \to \mathbb{E}^{\operatorname{Re}\varphi_2}_{S|\mathbb{P}}$. □

Recall that $C \subset \mathbb{C}^\times$ is a finite subset. The following result is a consequence of the preceding lemma. It shows how automorphisms of the Gaussian model on sectors can be interpreted as block matrices.

Proposition 2.9. *Let $S \subset \mathbb{P}$ be a sector (open or closed) at ∞, and assume that S is not a half-line whose direction is a Stokes direction for some $c, d \in C$. If we choose a numbering of the elements of C, i.e. $C = \{c_{(1)}, \dots, c_{(n)}\}$, we have*

$$\begin{aligned}
&\operatorname{Aut}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\Big(\pi^{-1}\mathbf{k}_S \otimes \bigoplus_{c\in C} \big(\mathbb{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{P}}\big)^{r_c}\Big) \simeq \operatorname{Aut}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\Big(\pi^{-1}\mathbf{k}_S \otimes \bigoplus_{c\in C} \big(\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{C}}\big)^{r_c}\Big)\\
&\simeq \big\{A = (A_{jk})_{j,k\in\{1,\dots,n\}} \in \mathbf{k}^{r\times r} \,\big|\, A_{jk} \in \mathbf{k}^{r_{c(j)}\times r_{c(k)}}, A_{jj} \text{ is invertible for any } j \in \{1,\dots,n\}\\
&\qquad\qquad \text{and } A_{jk} = 0 \text{ if } \operatorname{Re}\frac{c_{(j)}}{2}z^2 < \operatorname{Re}\frac{c_{(k)}}{2}z^2 \text{ for some } z \in S\big\}.
\end{aligned}$$

In particular, if $c_{(1)} <_S c_{(2)} < \ldots <_S c_{(n)}$, then the right hand side consists precisely of the invertible, lower block-triangular matrices with block sizes given by the numbers $r_{c_{(j)}}$.

Proof. By our assumption that S does not only consist of points on a Stokes line, we can assume the numbering to be chosen in such a way that $c_{(1)} <_{S'} c_{(2)} <_{S'} \ldots <_{S'} c_{(n)}$ on a subsector (at ∞) $S' \subseteq S$. Using Lemma 2.8, an endomorphism of $\pi^{-1}\mathbf{k}_S \otimes \bigoplus_{c\in C} \big(\mathbb{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{P}}\big)^{r_c}$ is identified with a lower block-triangular matrix whose block sizes are given by the $r_{c_{(j)}}$. Concretely, the block A_{jk} in the jth block row and the kth block column of A represents a morphism

$$\pi^{-1}\mathbf{k}_S \otimes \big(\mathbb{E}^{-\operatorname{Re}\frac{c_{(k)}}{2}z^2}_{\mathbb{C}|\mathbb{P}}\big)^{r_{c_{(k)}}} \longrightarrow \pi^{-1}\mathbf{k}_S \otimes \big(\mathbb{E}^{-\operatorname{Re}\frac{c_{(j)}}{2}z^2}_{\mathbb{C}|\mathbb{P}}\big)^{r_{c_{(j)}}}.$$

Such a matrix is invertible if and only if the blocks on the diagonal are invertible. □

Remark. It is important to observe the following: If we have an endomorphism of $\pi^{-1}\mathbf{k}_S \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$ which is represented by a matrix A, then the induced endomorphism of $\pi^{-1}\mathbf{k}_{S'} \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$ for a smaller sector $S' \subseteq S$ is represented by the same matrix A. Hence, given an endomorphism on S' such that the corresponding matrix satisfies the conditions necessary for endomorphisms on S, it extends to an endomorphism on S given by the same matrix. This also means that changing the radius of the sector S has no impact on the set of automorphisms.

Proposition 2.10. *Let $\mathcal{M}$ be of pure Gaussian type C. For any (open or closed) sector S at ∞ of sufficiently small radius intersecting at most one Stokes line for each pair $c, d \in C$, there is an isomorphism*

$$\pi^{-1}\mathbf{k}_S \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \pi^{-1}\mathbf{k}_S \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}. \tag{2.3}$$

Proof. By Lemma 2.2 (ii), for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have a small (i.e. with R big) open sector S_θ at ∞ for which there is such an isomorphism. Since $\mathbb{R}/2\pi\mathbb{Z}$ is compact, finitely many of the sectors S_θ suffice to cover all directions. In particular, we can choose a common radius for all these sectors, which we will assume for all sectors appearing in the rest of this proof.

Let S be an open sector at ∞ which intersects only one Stokes line for each pair $c, d \in C$. If $\#C = n$, there are at most $\binom{n}{2}$ Stokes lines intersecting S. We can write S as a finite union of narrower sectors $S_1, \dots, S_k$ on which the desired isomorphism holds, each of which contains at most one Stokes direction (for *some* pair $c, d \in C$), and such that each Stokes direction in S is contained in exactly one of these sectors. We may also assume that there is no inclusion $S_j \subseteq \bigcup_{l\neq j} S_l$ for any j.

Let us write for short $H := \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ and $\mathbb{M} := \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$. The following argument enables us to recursively obtain the desired isomorphism (2.3): Assume that we are given two open sectors $S_1, S_2 \subset S$ at ∞ with isomorphisms

$$\alpha_j\colon \pi^{-1}\mathbf{k}_{S_j} \otimes H \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S_j} \otimes \mathbb{M} \tag{2.4}$$

for $j \in \{1, 2\}$ and assume moreover that $S_1 \cap S_2 \neq \varnothing$, that we have $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$, that S_2 contains at most one Stokes direction and this Stokes direction (and also any other for the same pair c, d) is not contained in S_1 (see Fig. 2.2).

Choose a numbering of the elements of C such that $c_{(1)} <_{S_1\cap S_2} c_{(2)} <_{S_1\cap S_2} \cdots <_{S_1\cap S_2} c_{(n)}$. Applying the functor $\pi^{-1}\mathbf{k}_{S_1\cap S_2} \otimes (\bullet)$, the isomorphisms α_j induce two isomorphisms

$$\widetilde{\alpha}_j\colon \pi^{-1}\mathbf{k}_{S_1\cap S_2} \otimes H \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S_1\cap S_2} \otimes \mathbb{M}.$$

By Proposition 2.9, the transition isomorphism $\widetilde{\alpha}_2 \circ \widetilde{\alpha}_1^{-1}$ can be represented by a lower block-triangular matrix $A = (A_{jk})$. We can decompose A as follows:

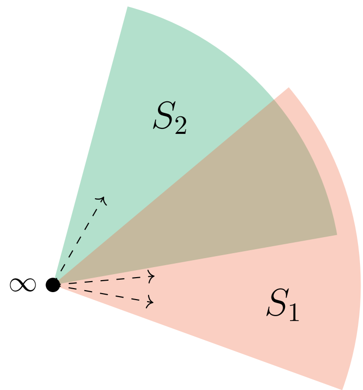


Figure 2.2.: The situation for gluing isomorphisms of two adjacent sectors. The arrows indicate possible Stokes directions.

- If S_2 contains a Stokes direction θ_2 for some pair of parameters:
  Note that this direction could be a Stokes direction for multiple pairs of parameters. We set $\mathcal{I} := \{(l, l') \mid l < l' \text{ and } \theta_2 \text{ is a Stokes direction for the pair } c_{(l)}, c_{(l')}\}$. Let A' be the block matrix (with the same block structure as A) having identity matrices on the diagonal and $A'_{l'l} = A^{-1}_{l'l'} A_{l'l}$ for any $(l, l') \in \mathcal{I}$. All the other blocks of A' are zero. Let A'' be the matrix obtained from A by subtracting the l'-th block column, multiplied by $A^{-1}_{l'l'} A_{l'l}$ from the right, from the l-th block column for each $(l, l') \in \mathcal{I}$ (working through $\mathcal{I}$ from high to low l'). Hence, A'' is still lower block-triangular and the entries $A''_{l'l}$ for $(l, l') \in \mathcal{I}$ are zero. It is easy to check that $A = A''A'$.

- If S_2 contains no Stokes direction at all:
  Set $A'' := A$ and $A' := I_r$. Obviously, $A = A''A'$.

It is not difficult to see that, in either of the two cases, the matrix A' represents an automorphism of $\pi^{-1}\mathbf{k}_{S_1} \otimes \mathbb{M}$ and the matrix A'' represents an automorphism of $\pi^{-1}\mathbf{k}_{S_2} \otimes \mathbb{M}$ (by the correspondence of Proposition 2.9), which we will also denote by A' and A''. The idea is now to use these matrices to perform additional "base changes" on the right hand sides of (2.4) in such a way that the transition isomorphism becomes the identity.

Consider the diagram

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{S_1 \cap S_2} \otimes H \arrow[r] \arrow[d, "A' \circ \widetilde{\alpha}_1"] & \pi^{-1}\mathbf{k}_{S_1} \otimes H \oplus \pi^{-1}\mathbf{k}_{S_2} \otimes H \arrow[r] \arrow[d, "A' \circ \alpha_1"', "A''^{-1} \circ \alpha_2"] & \pi^{-1}\mathbf{k}_{S_1 \cup S_2} \otimes H \arrow[r, "+1"] \arrow[d, dashed, "\widehat{\alpha}"] & {} \\
\pi^{-1}\mathbf{k}_{S_1 \cap S_2} \otimes \mathbb{M} \arrow[r] & \pi^{-1}\mathbf{k}_{S_1} \otimes \mathbb{M} \oplus \pi^{-1}\mathbf{k}_{S_2} \otimes \mathbb{M} \arrow[r] & \pi^{-1}\mathbf{k}_{S_1 \cup S_2} \otimes \mathbb{M} \arrow[r, "+1"] & {} ,
\end{tikzcd}$$

where the rows are distinguished triangles obtained by pulling back the standard exact sequence

$$0 \longrightarrow \mathbf{k}_{S_1 \cap S_2} \longrightarrow \mathbf{k}_{S_1} \oplus \mathbf{k}_{S_2} \longrightarrow \mathbf{k}_{S_1 \cup S_2} \longrightarrow 0$$

of sheaves (cf. [21, Proposition 2.3.6 (vii)]) via π^{-1} and tensoring with the enhanced ind-sheaves H and $\mathbb{M}$, respectively.

The vertical arrow in the middle denotes the morphism between the two direct sums given by the morphism $A' \circ \alpha_1$ between the first summands and the morphism $A'' \circ \alpha_2$ between the second summands (and zero morphisms from $\pi^{-1}\mathbf{k}_{S_1} \otimes H$ to $\pi^{-1}\mathbf{k}_{S_2} \otimes \mathbb{M}$ and from $\pi^{-1}\mathbf{k}_{S_2} \otimes H$ to $\pi^{-1}\mathbf{k}_{S_1} \otimes \mathbb{M}$). By our construction of A' and A'', the square on the left of the diagram commutes and the vertical arrows are isomorphisms. Therefore, there exists an isomorphism $\widehat{\alpha}$ completing the diagram to an isomorphism of distinguished triangles (cf. [25, Definition 10.1.16 and Proposition 10.1.15]). By Lemma 2.11 below and [25, Proposition 10.1.17], this isomorphism is even unique with the property of fitting into this diagram. (Of course, there is not a unique isomorphism $\pi^{-1}\mathbf{k}_{S_1\cup S_2} \otimes H \simeq \pi^{-1}\mathbf{k}_{S_1\cup S_2} \otimes \mathbb{M}$ as the target has nontrivial automorphisms, so also the desired isomorphism (2.3) is not unique.)

If S is a closed sector at ∞ (of sufficiently small radius) that intersects at most one Stokes line for each pair $c, d \in C$, then there exists an open sector with the same property containing S. For the latter, there is an isomorphism of the form (2.3). Tensoring it with $\pi^{-1}\mathbf{k}_S$ yields such an isomorphism for S. $\square$

Lemma 2.11. *For an open sector S at ∞, a proper open subsector $S' \subsetneq S$, and $c, d \in C$, one has*

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}, \pi^{-1}\mathbf{k}_{S'} \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{d}{2}z^2}\big) \simeq 0$$

and

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\pi^{-1}\mathbf{k}_{S'} \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}[1], \pi^{-1}\mathbf{k}_S \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{d}{2}z^2}\big) \simeq 0.$$

Proof. By arguments similar to those in the proof of Lemma 2.8, we obtain

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}, \pi^{-1}\mathbf{k}_{S'} \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{d}{2}z^2}\big)\\
&\qquad\simeq \varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \pi^{-1}\mathbf{k}_{S'} \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}}\big)
\end{aligned}$$

and this is always zero: We can apply the left exact functor

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \pi^{-1}\mathbf{k}_{S'} \otimes \bullet\,\big)$$

to the short exact sequence

$$0 \longrightarrow \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}\cap\{t-\mathrm{Re}\,\frac{c}{2}z^2<0\}} \longrightarrow \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}} \longrightarrow \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}\cap\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}} \longrightarrow 0.$$

It is easy to see (by considering stalks) that

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \pi^{-1}\mathbf{k}_{S'} \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}\cap\{t-\mathrm{Re}\,\frac{c}{2}z^2<0\}}\big) \simeq 0,$$

and therefore $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \pi^{-1}\mathbf{k}_{S'} \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}}\big)$ is isomorphic to a subspace of

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \pi^{-1}\mathbf{k}_{S'} \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}\cap\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}\big)\\
&\qquad\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \nu_!\nu^{-1}\pi^{-1}\mathbf{k}_{S'}\big)
\end{aligned}$$

$$\simeq \mathrm{Hom}_{\mathrm{D^b}(\mathbf{k}_N)}\big(\nu^{-1}(\pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}), \nu^{-1}(\pi^{-1}\mathbf{k}_{S'})\big)$$
$$\simeq \mathrm{Hom}_{\mathrm{D^b}(\mathbf{k}_N)}\big(\mathbf{k}_{(S\times\mathbb{R})\cap N}, \mathbf{k}_{(S'\times\mathbb{R})\cap N}\big) \simeq 0,$$

where we wrote for short $N := \{t - \mathrm{Re}\,\frac{d}{2}z^2 \geq a\} \cap \{t - \mathrm{Re}\,\frac{c}{2}z^2 \geq 0\}$, and $\nu\colon N \hookrightarrow \mathbb{C}\times\mathbb{R}$ denotes the (closed, hence proper) embedding. The last isomorphism follows from Lemma A.2 since $(S'\times\mathbb{R})\cap N \subset (S\times\mathbb{R})\cap N$ is a proper open (and not closed) subset.

For the second part, observe that

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{E^b}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\pi^{-1}\mathbf{k}_{S'} \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}[1], \pi^{-1}\mathbf{k}_S \otimes \mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{d}{2}z^2}\big)\\ &\qquad\simeq \varinjlim_{a\to\infty} \mathrm{Ext}^{-1}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_{S'} \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{c}{2}z^2\geq 0\}}, \pi^{-1}\mathbf{k}_S \otimes \mathbf{k}_{\{t-\mathrm{Re}\,\frac{d}{2}z^2\geq a\}}\big),\end{aligned}$$

which is zero since there are no Ext groups of negative degree (cf. [25, Proposition 13.1.10]). □

2.3. Stokes multipliers and monodromy

Let $C \subset \mathbb{C}^\times$ be a finite subset and $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be of pure Gaussian type C. As we have seen, for any pair of parameters $c, d \in C$ there are exactly four Stokes directions, which differ by multiples of $\frac{\pi}{2}$. Therefore, we generally need four sectors to cover a neighbourhood of ∞ by sectors on which we have isomorphisms of the type (2.3).

Fix a generic direction θ_0, and choose a numbering of the elements of C such that $c_{(1)} <_{\theta_0} c_{(2)} <_{\theta_0} \ldots <_{\theta_0} c_{(n)}$. Clearly, $\theta_0 + k\frac{\pi}{2}$ (for $k \in \{1,2,3\}$) are also generic. By Proposition 2.10, there exists $R \in \mathbb{R}_{>0}$ such that on the closed sectors[4]

$$\Sigma_k := \left\{ z \in \mathbb{C} \,\middle|\, |z| \geq R, \arg z \in \left[\theta_0 + (k-1)\frac{\pi}{2}, \theta_0 + k\frac{\pi}{2}\right]\right\} \qquad (k \in \mathbb{Z}/4\mathbb{Z})$$

we have isomorphisms

$$\alpha_k\colon \pi^{-1}\mathbf{k}_{\Sigma_k} \otimes \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}(\mathcal{M}) \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{\Sigma_k} \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}\big)^{r_c}. \tag{2.5}$$

(Note that these isomorphisms are not unique, so this step involves a choice.)

On the half-lines $\Sigma_{k-1,k} := \Sigma_{k-1} \cap \Sigma_k$ and $\Sigma_{k,k+1} := \Sigma_k \cap \Sigma_{k+1}$ bounding the sector Σ_k, α_k induces isomorphisms

$$\alpha_k^{k-1}\colon \pi^{-1}\mathbf{k}_{\Sigma_{k-1,k}} \otimes \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}(\mathcal{M}) \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{\Sigma_{k-1,k}} \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}\big)^{r_c}$$

and

$$\alpha_k^{k+1}\colon \pi^{-1}\mathbf{k}_{\Sigma_{k,k+1}} \otimes \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}(\mathcal{M}) \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{\Sigma_{k,k+1}} \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\mathrm{Re}\,\frac{c}{2}z^2}\big)^{r_c}.$$

[4]Throughout this thesis, we will often denote the elements of $\mathbb{Z}/4\mathbb{Z}$ by 1, 2, 3 and 4 (instead of 0) in order to reflect the intuitive numbering of sectors and quadrants by numbers between 1 and 4.

Hence, we get a transition isomorphism on each $\Sigma_{k,k+1}$ given by

$$\alpha_{k+1}^{k} \circ (\alpha_{k}^{k+1})^{-1} \in \mathrm{Aut}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\Big(\pi^{-1}\mathbf{k}_{\Sigma_{k,k+1}} \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}\Big),$$

which is represented by an invertible, block-triangular matrix σ_k (cf. Proposition 2.9). It is upper block-triangular for $k \in \{1, 3\}$ and lower block-triangular for $k \in \{2, 4\}$.

Definition 2.12. The matrices σ_k are called *Stokes multipliers* (or *Stokes matrices*) and their product $\sigma_4\sigma_3\sigma_2\sigma_1$ is called *(topological) monodromy* of $\mathcal{M}$ (with respect to θ_0).

Note that these notions require fixing a generic direction.

Proposition 2.13. *The monodromy of a D-module of pure Gaussian type is the identity, i.e.* $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$.

Proof. Choose $\rho > R$ and set $B := \{z \in \mathbb{C} \mid |z| \leq \rho\}$. By Lemma 1.4, there is a canonical isomorphism

$$\tau\colon \pi^{-1}\mathbf{k}_B \otimes \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_B \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r.$$

We introduce the notation $D_k := \Sigma_k \cap B$ and $D_{k,k+1} := D_k \cap D_k$ and set $D := \bigcup_{k\in\mathbb{Z}/4\mathbb{Z}} D_k$ (see Fig. 2.3). Moreover, we write for short $H := \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ and $\mathbb{M} := \bigoplus_{c\in C} \big(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$.

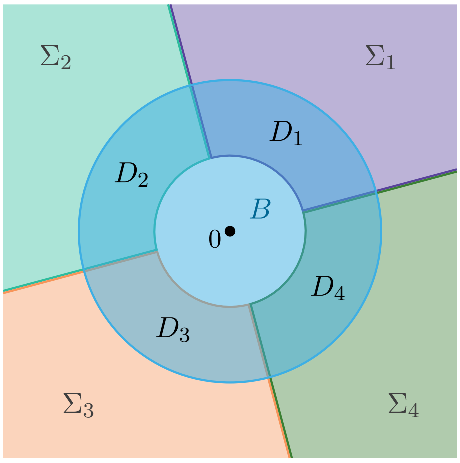


Figure 2.3.: The proof of Proposition 2.13 compares the situation around 0 with the situation around ∞: The sets D_k are the overlaps of the four sectors Σ_k at ∞ with the circle B around the origin.

Consider the distinguished triangle

$$\pi^{-1}\mathbf{k}_{D_1\cup D_2} \otimes H \longrightarrow \pi^{-1}\mathbf{k}_{D_1} \otimes H \oplus \pi^{-1}\mathbf{k}_{D_2} \otimes H \longrightarrow \pi^{-1}\mathbf{k}_{D_{12}} \otimes H \xrightarrow{+1} .$$

By using the isomorphisms induced by α_1, α_2 and α_2^1, we obtain an isomorphic distinguished triangle

$$\pi^{-1}\mathbf{k}_{D_1\cup D_2}\otimes H \longrightarrow \pi^{-1}\mathbf{k}_{D_1}\otimes\mathbb{M}\oplus\pi^{-1}\mathbf{k}_{D_2}\otimes\mathbb{M}\longrightarrow\pi^{-1}\mathbf{k}_{D_{12}}\otimes\mathbb{M}\xrightarrow{+1},$$

where the second morphism is given as follows: The morphism $\pi^{-1}\mathbf{k}_{D_1}\otimes\mathbb{M}\to\pi^{-1}\mathbf{k}_{D_{12}}\otimes\mathbb{M}$ is the canonical morphism composed with the automorphism induced by $\alpha_2^1\circ(\alpha_1^2)^{-1}$, i.e. it is given by the matrix σ_1. The morphism $\pi^{-1}\mathbf{k}_{D_2}\otimes\mathbb{M}\to\pi^{-1}\mathbf{k}_{D_{12}}\otimes\mathbb{M}$ is the negative of the canonical morphism.

Using the isomorphism τ, we get a diagram

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{D_1\cup D_2}\otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{D_1}\otimes\mathbb{M}\oplus\pi^{-1}\mathbf{k}_{D_2}\otimes\mathbb{M} \arrow[r] \arrow[d, "\simeq"] & \pi^{-1}\mathbf{k}_{D_{12}}\otimes\mathbb{M} \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\
\pi^{-1}\mathcal{G}_{12}\otimes\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \arrow[r] & \pi^{-1}\mathbf{k}_{D_1}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r\oplus\pi^{-1}\mathbf{k}_{D_2}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \arrow[r] & \pi^{-1}\mathbf{k}_{D_{12}}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \arrow[r, "+1"] & {}
\end{tikzcd} \tag{2.6}$$

(We start with the black part of this diagram.) Since the isomorphism τ is essentially defined by the identity matrix (see the proof of Lemma 1.4), the arrow in the second line must be given similarly to the corresponding one in the first line: The morphism $\pi^{-1}\mathbf{k}_{D_1}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r\to\pi^{-1}\mathbf{k}_{D_{12}}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r$ is represented by the matrix σ_1, and the morphism $\pi^{-1}\mathbf{k}_{D_2}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r\to\pi^{-1}\mathbf{k}_{D_{12}}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r$ is the negative of the canonical one.

Now consider the kernel of the morphism of sheaves on $\mathbb{P}$

$$(\mathbf{k}_{D_1})^r\oplus(\mathbf{k}_{D_2})^r\xrightarrow{\sigma_1-\mathbb{1}}(\mathbf{k}_{D_{12}})^r.$$

It is a local system on $D_1\cup D_2$ (extended by zero), glued from two constant sheaves of rank r via the gluing matrix σ_1. Let us denote this kernel by $\mathcal{G}_{12}$. Since $\pi^{-1}\mathbf{k}_{D_k}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r\simeq\pi^{-1}(\mathbf{k}_{D_k})^r\otimes\mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$, we can complete the second line of (2.6) as shown in blue colour, and we get an isomorphism $\pi^{-1}\mathbf{k}_{D_1\cup D_2}\otimes H\simeq\pi^{-1}\mathcal{G}_{12}\otimes\mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$.

Let us remark that this isomorphism is unique with the property of fitting into the diagram since one can easily check the conditions of [11, Corollary IV.1.5]: We have

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\pi^{-1}\mathcal{G}_{12}\otimes\mathbf{k}_{\mathbb{P}}^{\mathrm{E}},\pi^{-1}\mathbf{k}_{D_{12}}\otimes(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r[-1]\big)\simeq\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}})}\big(\mathcal{G}_{12},(\mathbf{k}_{D_{12}})^r[-1]\big)\simeq 0,$$

where the first isomorphism follows from the full faithfulness of the functor $\pi^{-1}(\bullet)\otimes\mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$ (see [6, Proposition 4.7.15]) and the second one follows from the vanishing of negative-degree Ext groups (see [11, Theorem III.5.5]). Uniqueness arguments in the remainder of the proof are analogous.

One can construct in an analogous way the sheaf $\mathcal{G}_{34}$ glued from two constant sheaves on D_3 and D_4 via the gluing matrix σ_3, as well as an isomorphism $\pi^{-1}\mathbf{k}_{D_3\cup D_4}\otimes H\simeq\pi^{-1}\mathcal{G}_{34}\otimes\mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$.

Finally, we consider the diagram (starting again with the black part)

$$\begin{tikzcd} \pi^{-1}\mathbf{k}_D \otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{D_1\cup D_2} \otimes H \oplus \pi^{-1}\mathbf{k}_{D_3\cup D_4} \otimes H \arrow[r] \arrow[d, "\simeq"] & \pi^{-1}\mathbf{k}_{D_{41}\cup D_{23}} \otimes H \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\ \pi^{-1}\mathcal{G} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \arrow[r] & \pi^{-1}\mathcal{G}_{12} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \oplus \pi^{-1}\mathcal{G}_{34} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \arrow[r] & \pi^{-1}\mathbf{k}_{D_{41}\cup D_{23}} \otimes \pi^{-1}\mathcal{G}_{34} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \arrow[r, "+1"] & {} \end{tikzcd}, \tag{2.7}$$

where the vertical isomorphisms are given by the ones constructed above. In particular, the rightmost vertical isomorphism is induced by $\pi^{-1}\mathbf{k}_{D_3\cup D_4} \otimes H \simeq \pi^{-1}\mathcal{G}_{34} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$.

The sheaf $\mathcal{G}$ defining the blue object which completes the diagram is given as follows: It is easy to see from the definitions that there are isomorphisms $\mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{12} \simeq (\mathbf{k}_{D_{41}})^r \oplus (\mathbf{k}_{D_{23}})^r$ and $\mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{34} \simeq (\mathbf{k}_{D_{41}})^r \oplus (\mathbf{k}_{D_{23}})^r$, and hence one can define an isomorphism $\mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{12} \xrightarrow{\sim} \mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{34}$ given by the matrices σ_4^{-1} and σ_2 on the first and second summand, respectively. Let $\mathcal{G}$ be the kernel of the morphism

$$\mathcal{G}_{12} \oplus \mathcal{G}_{34} \longrightarrow \mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{34}, \tag{2.8}$$

where $\mathcal{G}_{12} \to \mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{34}$ is given by the composition of the canonical morphism $\mathcal{G}_{12} \to \mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{12}$ and the isomorphism just described. The morphism $\mathcal{G}_{34} \to \mathbf{k}_{D_{41}\cup D_{23}} \otimes \mathcal{G}_{34}$ is the negative of the canonical one. We remark that the morphism (2.8) is exactly the one that induces the second morphism in the lower row of (2.7), and hence we can complete the diagram with the blue part.

In this way, we have shown that there is an isomorphism $\pi^{-1}\mathbf{k}_D \otimes H \simeq \pi^{-1}\mathcal{G} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$. The sheaf $\mathcal{G}$ is a local system on D (extended by zero to $\mathbb{P}$), glued from constant sheaves on the D_k with gluing matrices σ_k. Consequently, its monodromy (choosing a base point in D_1) is the product $\sigma_4\sigma_3\sigma_2\sigma_1$. On the other hand, by Lemma 2.2, we have an isomorphism $\pi^{-1}\mathbf{k}_D \otimes H \simeq \pi^{-1}(\mathbf{k}_D)^r \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$. Since by [6, Proposition 4.7.15] the functor $\pi^{-1}(\bullet) \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$ is fully faithful, this means that $\mathcal{G}$ is isomorphic to the constant local system and hence that their monodromies are equal. $\square$

2.4. Closed sectors of infinite radius

The question studied in this section is how large we can choose the radius of the four sectors. It will turn out that the absence of singularities of a D-module of pure Gaussian type outside the point ∞ enables us to increase the sectors' radii as far as we like. Hence, we can actually use sectors of infinite radius.

Let $\mathcal{M}$ be of pure Gaussian type and recall the generic direction θ_0 and the Stokes multipliers σ_k from the previous section. Write $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ and $\mathbb{r} = (r_c)_{c\in C}$.

We define the sectors

$$S_k := \left\{ z \in \mathbb{C} \,\middle|\, \arg z \in \left[\theta_0 + (k-1)\frac{\pi}{2}, \theta_0 + k\frac{\pi}{2}\right] \text{ if } z \neq 0 \right\},$$

which are closed sectors of infinite radius at ∞, but can also be considered as closed sectors (including the vertex) at 0. As usual, we set $S_{k,k+1} := S_k \cap S_{k+1}$.

Definition 2.14. We define the enhanced sheaf $\mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}} \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$ (or $\mathcal{F}_\sigma$ for short) on $\mathbb{C}$ by the following data (recall Notation 1.2):

- $(\mathcal{F}_\sigma)_{S_k}$ is isomorphic to $\pi^{-1}\mathbf{k}_{S_k} \otimes \bigoplus\limits_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$ for any $k \in \mathbb{Z}/4\mathbb{Z}$.
- For all $k \in \mathbb{Z}/4\mathbb{Z}$, the induced transition isomorphisms, which are automorphisms of $\pi^{-1}\mathbf{k}_{S_{k,k+1}} \otimes \bigoplus\limits_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$, are given by the matrices σ_k.

It follows from Lemma A.6 together with Proposition 2.13 that this defines – uniquely up to unique isomorphism – a sheaf $\mathcal{F}_\sigma \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$. If an enhanced sheaf on $\mathbb{C}$ is isomorphic to one of this form for suitable data C, θ_0, $\mathbb{r}$ and σ, we call it an *enhanced sheaf of pure Gaussian type*.

Remark. Let us briefly reflect on the meaning of "suitable data" in the above definition. This expression is to say that $C \subset \mathbb{C}^\times$ is a finite subset, $\mathbb{r}$ is a family of positive integers (one for each element of C) and, most importantly, $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$ is a generic direction with respect to C. It is not difficult to see that an enhanced sheaf $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$ is of pure Gaussian type if and only if it satisfies the first condition in Definition 2.14. Then, in view of Proposition 2.9, the gluing data will automatically be given by four "suitable" matrices, i.e. ones which satisfy the conditions of Stokes multipliers for a D-module of pure Gaussian type. These conditions are collected in Definition 2.18 later (conditions on an object of the category of Stokes data).

To our D-module $\mathcal{M}$ of pure Gaussian type we can attach an enhanced sheaf of pure Gaussian type $\mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}$ using the data from the previous section. The following theorem shows that this finally is an enhanced sheaf (on $\mathbb{C}$) describing globally (on $\mathbb{P}$) the enhanced solutions of $\mathcal{M}$. Note that, in contrast to the formulation of Lemma 2.3, we do not write extension by zero.

Theorem 2.15. *There is an isomorphism*

$$\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}.$$

In the proof of this theorem, let us write $\mathcal{F} := \mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}$. We will make use of the following lemma, which gives an alternative description of $\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ away from the singularity.

Lemma 2.16. *Let $B \subset \mathbb{C}$ be a closed ball of finite radius around 0. Then there is an isomorphism*

$$\pi^{-1}\mathbf{k}_B \otimes \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M}) \simeq \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}. \tag{2.9}$$

Proof. We abbreviate $B_k := B \cap S_k$ and $B_{k,k+1} := B \cap S_{k,k+1}$ for $k \in \mathbb{Z}/4\mathbb{Z}$, as well as $H := \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ and $\mathsf{M} := \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$. Note that the extension by zero to $\mathbb{P}$ of the latter is $\tilde{i}_! \mathsf{M} \simeq \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$, and we will (as usual) suppress the functor $\tilde{i}_!$. Recall moreover that we write for short $\mathcal{F}_Z$ instead of $\pi^{-1}\mathbf{k}_Z \otimes \mathcal{F}$.

From Lemma 2.2, we have an isomorphism

$$\vartheta \colon \pi^{-1}\mathbf{k}_B \otimes H \xrightarrow{\sim} \pi^{-1}\mathbf{k}_B \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r,$$

and hence a distinguished triangle (whose second morphism is the difference of the two canonical ones)

$$\pi^{-1}\mathbf{k}_{B_1\cup B_2} \otimes H \longrightarrow \pi^{-1}\mathbf{k}_{B_1} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \oplus \pi^{-1}\mathbf{k}_{B_2} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \longrightarrow \pi^{-1}\mathbf{k}_{B_{12}} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \xrightarrow{+1} .$$

Using the canonical isomorphism

$$\tau \colon \pi^{-1}\mathbf{k}_B \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \xrightarrow{\sim} \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}$$

coming from Lemma 1.4 and the induced isomorphisms τ_k, $\tau_{k,k+1}$ on B_k, $B_{k,k+1}$, we obtain an isomorphism as the blue vertical arrow in the diagram

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{B_1\cup B_2} \otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{B_1} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \oplus \pi^{-1}\mathbf{k}_{B_2} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \arrow[r] \arrow[d, "\tau_1"', "\tau_2\circ\sigma_1"] & \pi^{-1}\mathbf{k}_{B_{12}} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \arrow[r, "+1"] \arrow[d, "\tau_{12}\circ\sigma_1"] & {} \\
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_1\cup B_2} \arrow[r] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_1} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_2} \arrow[r, "\sigma_1-\mathbb{1}"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_{12}} \arrow[r, "+1"] & {} .
\end{tikzcd}$$

Here, the σ_1 in the vertical arrows denotes the automorphism of $(\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \simeq \pi^{-1}(\mathbf{k}_{\mathbb{P}})^r \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$ induced by the automorphism of the local system $(\mathbf{k}_{\mathbb{P}})^r$ which is given by the matrix σ_1. Note that, by a statement analogous to Lemma 2.11 (for closed subsets $S \subsetneq S' \subseteq \mathbb{C}$) together with [25, Proposition 10.1.17], the blue arrow in the above diagram is unique.

Similarly, we have the diagram

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{B_3\cup B_4} \otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{B_4} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \oplus \pi^{-1}\mathbf{k}_{B_3} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \arrow[r] \arrow[d, "\tau_4\circ\sigma_3\sigma_2\sigma_1"', "\tau_3\circ\sigma_2\sigma_1"] & \pi^{-1}\mathbf{k}_{B_{34}} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \arrow[r, "+1"] \arrow[d, "\tau_{34}\circ\sigma_3\sigma_2\sigma_1"] & {} \\
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_3\cup B_4} \arrow[r] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_4} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_3} \arrow[r, "\mathbb{1}-\sigma_3"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_{34}} \arrow[r, "+1"] & {} .
\end{tikzcd}$$

Finally, the desired isomorphism is obtained from

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{B} \otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{B_1\cup B_2} \otimes H \oplus \pi^{-1}\mathbf{k}_{B_3\cup B_4} \otimes H \arrow[r] \arrow[d, "\simeq"] & \pi^{-1}\mathbf{k}_{B_{41}\cup B_{23}} \otimes H \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B} \arrow[r] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_1\cup B_2} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_3\cup B_4} \arrow[r, "\gamma\circ\mathrm{can}-\mathrm{can}"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_{41}\cup B_{23}} \arrow[r, "+1"] & {} ,
\end{tikzcd}$$

where the black vertical arrows are induced by the ones constructed above. In the second row, "can" denotes canonical morphisms and γ is the morphism fitting into the diagram (whose rows come from the second rows of the diagrams above)

$$\begin{tikzcd}
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_{41}\cup B_{23}} \arrow[r] \arrow[d, dashed, "\gamma"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_{41}} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_{23}} \arrow[r, "\sigma_1-\mathbb{1}"] \arrow[d, "\sigma_4^{-1}"', "\sigma_2"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{\{0\}} \arrow[r, "+1"] \arrow[d, "\sigma_3\sigma_2"] & {} \\
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{B_{41}\cup B_{23}} \arrow[r] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_{41}} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{B_{23}} \arrow[r, "\mathbb{1}-\sigma_3"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{\{0\}} \arrow[r, "+1"] & {} .
\end{tikzcd}$$

Note that this diagram only commutes since the monodromy is trivial (i.e. $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$).

One can even show that the blue isomorphism $\pi^{-1}\mathbf{k}_B \otimes H \simeq \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}$ is uniquely determined by the diagram above, checking the assumption of [11, Corollary IV.1.5]: For closed subsets $B' \subsetneq B \subseteq \mathbb{C}$, we have (similarly to the proof of Lemma 2.8)

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}\big(\pi^{-1}\mathbf{k}_{B'} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}, \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}[-1]\big)\\
&\qquad \simeq \varinjlim_{a\to\infty} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_{B'} \otimes \mathcal{F}, \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\{t\geq a\}} \overset{*}{\otimes} \mathcal{F}[-1]\big)\\
&\qquad \simeq \varinjlim_{a\to\infty} \mathrm{Ext}^{-1}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})}\big(\pi^{-1}\mathbf{k}_{B'} \otimes \mathcal{F}, \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\{t\geq a\}} \overset{*}{\otimes} \mathcal{F}\big) \simeq 0.
\end{aligned}$$

For the second isomorphism, note that $\mathbf{k}_{\{t\geq a\}} \overset{*}{\otimes} \mathcal{F}$ is concentrated in degree 0, which follows easily from Lemma A.5. The last isomorphism is then clear since Ext groups of negative degree vanish (see [11, Theorem III.5.5]). $\square$

Proof of Theorem 2.15. We abbreviate $H := \mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}(\mathcal{M})$ and $\mathsf{M} := \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$. Moreover, we choose $\rho > R$ and set $B := \{z \in \mathbb{C} \mid |z| \leq \rho\}$, $\Sigma := \bigcup_{k\in\mathbb{Z}/4\mathbb{Z}} \Sigma_k$, $D := B \cap \Sigma$, $D_k := D \cap \Sigma_k$ and $D_{k,k+1} := D \cap \Sigma_{k,k+1}$).

Uniqueness arguments for the arrows completing morphisms of distinguished triangles in this proof work as in the proof of Lemma 2.16, and we will not repeat them here.

Firstly, we use distinguished triangles to obtain an isomorphism

$$\pi^{-1}\mathbf{k}_\Sigma \otimes H \simeq \pi^{-1}\mathbf{k}_\Sigma \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}. \tag{2.10}$$

In order to do this, we start with the following diagram, whose rows are distinguished triangles:

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{\Sigma_1\cup\Sigma_2} \otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{\Sigma_1} \otimes H \oplus \pi^{-1}\mathbf{k}_{\Sigma_2} \otimes H \arrow[r] \arrow[d, "\alpha_1"', "\alpha_2"] & \pi^{-1}\mathbf{k}_{\Sigma_{12}} \otimes H \arrow[r, "+1"] \arrow[d, "\alpha_2^1"] & {} \\
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{\Sigma_1\cup\Sigma_2} \arrow[r] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{\Sigma_1} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{\Sigma_2} \arrow[r, "\sigma_1-\mathbb{1}"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}_{\Sigma_{12}} \arrow[r, "+1"] & {} .
\end{tikzcd}$$

By the construction of $\mathcal{F}$, the second row is indeed a distinguished triangle and we obtain the blue, vertical isomorphism. Proceeding with further diagrams of this type, analogously to the proof of Lemma 2.16, finally yields the desired isomorphism (2.10).

Secondly, we determine an isomorphism

$$\pi^{-1}\mathbf{k}_B \otimes H \simeq \pi^{-1}\mathbf{k}_B \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}. \tag{2.11}$$

The existence of such an isomorphism was shown in Lemma 2.16. However, it is important to note that such an isomorphism is neither canonical nor unique, but depends in particular – as can be seen from the proof – on the choice of a trivialization $\vartheta\colon \pi^{-1}\mathbf{k}_B \otimes H \simeq \pi^{-1}\mathbf{k}_B \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r$. Denote by ϑ_k and $\vartheta_{k,k+1}$ the induced isomorphisms obtained by tensoring with $\pi^{-1}\mathbf{k}_{D_k}$ and $\pi^{-1}\mathbf{k}_{D_{k,k+1}}$, respectively. We choose ϑ in such a way that the composition

$$\vartheta_1 \circ \alpha_1^{-1}\colon \pi^{-1}\mathbf{k}_{D_1} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{D_1} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \tag{2.12}$$

is the canonical one, which can be achieved by composing ϑ with an automorphism of the right-hand side. The rest of the construction of (2.11) is done as in the proof of Lemma 2.16.

Now consider the following diagram:

$$\begin{tikzcd}
\pi^{-1}\mathbf{k}_{\mathbb{C}} \otimes H \arrow[r] \arrow[d, dashed, "\simeq"] & \pi^{-1}\mathbf{k}_{\Sigma} \otimes H \oplus \pi^{-1}\mathbf{k}_B \otimes H \arrow[r] \arrow[d, "(2.10)"', "(2.11)"] & \pi^{-1}\mathbf{k}_D \otimes H \arrow[r, "+1"] \arrow[d, "(2.11)"] & {} \\
\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F} \arrow[r] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{\Sigma} \oplus \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_B \arrow[r, "\delta\circ\mathrm{can}-\mathrm{can}"] & \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_D \arrow[r, "+1"] & {} .
\end{tikzcd}$$

To see that the blue object completes the triangle in the second line, one needs to understand the automorphism δ of $\pi^{-1}\mathbf{k}_D \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}$. It is obtained by composing the inverse of the isomorphism induced by (2.10) with the isomorphism induced by (2.11). By the constructions of these isomorphisms, one can see that, tensoring δ with $\pi^{-1}\mathbf{k}_{D_1}$, one obtains the morphism

$$\pi^{-1}\mathbf{k}_{D_1} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M} \xrightarrow{\alpha_1^{-1}} \pi^{-1}\mathbf{k}_{D_1} \otimes H \xrightarrow{\vartheta_1} \pi^{-1}\mathbf{k}_{D_1} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \xrightarrow{\tau_1} \pi^{-1}\mathbf{k}_{D_1} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}.$$

Recalling the notation τ_k from the proof of Lemma 2.16 and the fact that τ_k is induced by the identity matrix, it follows that this composition is the identity.

Similarly, tensoring δ with $\pi^{-1}\mathbf{k}_{D_2}$ yields the composition

$$\pi^{-1}\mathbf{k}_{D_2} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M} \xrightarrow{\alpha_2^{-1}} \pi^{-1}\mathbf{k}_{D_2} \otimes H \xrightarrow{\vartheta_2} \pi^{-1}\mathbf{k}_{D_2} \otimes (\mathbf{k}_{\mathbb{P}}^{\mathrm{E}})^r \xrightarrow{\tau_2\circ\sigma_1} \pi^{-1}\mathbf{k}_{D_2} \otimes \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{M}.$$

The morphism $\vartheta_2 \circ \alpha_2^{-1}$ is given by a matrix. Tensoring with $\pi^{-1}\mathbf{k}_{D_{12}}$, we get the morphism $\vartheta_{12} \circ (\alpha_2^1)^{-1} = \vartheta_{12} \circ (\alpha_1^2)^{-1} \circ \alpha_1^2 \circ (\alpha_2^1)^{-1}$, and this is represented by the matrix σ_1^{-1} due to (2.12). Therefore, $\pi^{-1}\mathbf{k}_{D_2} \otimes \delta$ is also the identity. Similarly, δ is the identity also on D_3 and D_4 and therefore $\delta = \mathrm{id}$ (noting that the distinguished triangles used for gluing δ from the $\pi^{-1}\mathbf{k}_{D_k} \otimes \delta$ satisfy the conditions of [25, Proposition 10.1.17] and hence the gluing is unique).

Since $\pi^{-1}\mathbf{k}_{\mathbb{C}} \otimes H \simeq H$, this concludes the proof of Theorem 2.15. □

With this result, we have reduced the description of the enhanced solutions for a D-module of pure Gaussian type to a small set of data. This will be useful for our calculation of the Fourier–Laplace transform. In the next section, we use this data to establish a Riemann–Hilbert correspondence for D-modules of pure Gaussian type.

2.5. Stokes data and a Riemann–Hilbert correspondence

We fix a finite subset $C \subset \mathbb{C}^\times$ as well as a generic direction θ_0 and consider the sectors $S_k = \{z \in \mathbb{C} \mid \arg z \in [\theta_0 + (k-1)\frac{\pi}{2}, \theta_0 + k\frac{\pi}{2}] \text{ if } z \neq 0\}$. We also fix a positive integer r_c for any $c \in C$.

Let $\mathrm{Mod}^\star_{\text{Gauß}}(\mathcal{D}_\mathbb{P})$ be the full subcategory of $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_\mathbb{P})$ consisting of objects of pure Gaussian type C and with a Levelt–Turrittin decomposition satisfying $\operatorname{rk} \mathcal{R}_c = r_c$ for every $c \in C$.

Let $\mathrm{E}^\star_{\text{Gauß}}(\mathrm{I}\mathbf{k}_\mathbb{P})$ be the full subcategory of $\mathrm{E}^\mathrm{b}(\mathrm{I}\mathbf{k}_\mathbb{P})$ consisting of objects H satisfying $\pi^{-1}\mathbf{k}_\mathbb{C} \otimes H \simeq H$ and admitting isomorphisms

$$\pi^{-1}\mathbf{k}_{S_k} \otimes H \simeq \pi^{-1}\mathbf{k}_{S_k} \otimes \bigoplus_{c\in C} \left(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\right)^{r_c} \tag{2.13}$$

for $k \in \mathbb{Z}/4\mathbb{Z}$. (Note that these isomorphisms are not part of the data.)

Proposition 2.17. *The functor $\mathcal{S}ol^\mathrm{E}_\mathbb{P}$ induces an equivalence between $\mathrm{Mod}^\star_{\text{Gauß}}(\mathcal{D}_\mathbb{P})$ and $\mathrm{E}^\star_{\text{Gauß}}(\mathrm{I}\mathbf{k}_\mathbb{P})$.*

Proof. The functor is well-defined by Theorem 2.15. Full faithfulness follows from the Riemann–Hilbert correspondence of [6] (see Theorem 1.10).

It remains to show that the functor is essentially surjective: Let $H \in \mathrm{E}^\star_{\text{Gauß}}(\mathrm{I}\mathbf{k}_\mathbb{P})$. Denote by $U_\infty := \mathbb{P} \smallsetminus \overline{B}_{\frac{1}{2}}(0)$ the neighbourhood of ∞ given by excluding the closed ball of radius $\frac{1}{2}$ around 0 (i.e. in the chart around ∞ with local coordinate z^{-1}, U_∞ is the open ball of radius 2). Denote by $U_0 := B_2(0)$ the neighbourhood of 0 given by the open ball of radius 2 around 0. Denote further by $j_0 \colon U_0 \hookrightarrow \mathbb{P}$ and $j_\infty \colon U_\infty \hookrightarrow \mathbb{P}$ the inclusions.

Since U_∞ is biholomorphic to the unit disc Δ, it follows from [32, Lemma 9.8] that there exists a meromorphic connection $\mathcal{M}_\infty \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{U_\infty})$ with pole at ∞ such that $\mathrm{E}j_\infty^{-1}H \simeq \mathcal{S}ol^\mathrm{E}_{U_\infty}(\mathcal{M}_\infty)$.

If we choose isomorphisms $\pi^{-1}\mathbf{k}_{S_k} \otimes H \simeq \pi^{-1}\mathbf{k}_{S_k} \otimes \bigoplus_{c\in C} \left(\mathbb{E}_{\mathbb{C}|\mathbb{P}}^{-\operatorname{Re}\frac{c}{2}z^2}\right)^{r_c}$ for $k \in \mathbb{Z}/4\mathbb{Z}$, we can define transition matrices σ_k, which will satisfy $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$ due to the compatibility at the origin (which lies in every sector S_k). Similarly to the arguments in the previous section, we get an isomorphism $H \simeq \mathbf{k}^\mathrm{E}_\mathbb{P} \overset{+}{\otimes} \mathcal{F}_\sigma$. In view of Lemma 2.16, this yields an isomorphism $\mathrm{E}j_0^{-1}H \simeq (\mathbf{k}^\mathrm{E}_{U_0})^r \simeq \mathcal{S}ol^\mathrm{E}_{U_0}(\mathcal{O}^r_{U_0})$.

From the identity morphism of H, we obtain an isomorphism $\mathcal{M}_\infty|_{U_0\cap U_\infty} \simeq \mathcal{O}^r_{U_0}|_{U_0\cap U_\infty}$ through the full faithfulness of $\mathcal{S}ol^\mathrm{E}_{U_0\cap U_\infty}$. As D-modules form a stack (cf. [25, Proposition 19.4.7]), we can glue $\mathcal{M}_\infty$ and $\mathcal{O}^r_{U_0}$ to obtain $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\mathbb{P})$. It is easy to check that it is holonomic and a meromorphic connection with pole at ∞. The Levelt–Turrittin decomposition at ∞ of $\mathcal{M}$ coincides with that of $\mathcal{M}_\infty$, and therefore the induced local description of $\mathcal{S}ol^{\mathrm{E}}_{U_\infty}(\mathcal{M}_\infty)$ on small sectors (cf. Proposition 1.12) must be isomorphic to the one given by (2.13). Applying [7, Corollary 5.2.3] (see also [32, Lemma 5.15]), which basically states that exponential factors and ranks are unique, this implies that $\mathcal{M} \in \mathrm{Mod}^\star_{\text{Gauß}}(\mathcal{D}_{\mathbb{P}})$.

Finally, since $U \mapsto \mathrm{E}^0(\mathrm{I}\mathbf{k}_U)$ is a stack (cf. [8, Proposition 2.6.4]) and $H \in \mathrm{E}^0(\mathrm{I}\mathbf{k}_{\mathbb{P}})$, we conclude $H \simeq \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}(\mathcal{M})$. □

A similar equivalence can, for example, be proved in the case where only C and θ_0 are fixed, but not the ranks r_c.

The results of the previous sections enable us to describe the objects of $\mathrm{E}^\star_{\text{Gauß}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})$ in terms of linear algebra data.

Definition 2.18. Let $C \subset \mathbb{C}^\times$ be a finite subset, θ_0 a generic direction with respect to C and $r_c \in \mathbb{Z}_{>0}$ for any $c \in C$. Choose a numbering of the elements of C such that $c_{(1)} <_{\theta_0} c_{(2)} <_{\theta_0} \ldots <_{\theta_0} c_{(n)}$. We will write r_j instead of $r_{c_{(j)}}$.

The *category* $\mathfrak{S}\mathfrak{D}^\star$ *of Stokes data of pure Gaussian type* $(C, \theta_0, (r_c)_{c\in C})$ is defined as follows:

- An object $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}} \in \mathrm{Ob}\,\mathfrak{S}\mathfrak{D}^\star$ is a family of four block matrices with the properties:
  - The block structure is given by the numbers r_j ($j \in \{1, \ldots, n\}$), i.e. the jth diagonal block has size $r_j \times r_j$.
  - The matrices σ_1 and σ_3 are upper block-triangular and the matrices σ_2 and σ_4 are lower block-triangular.
  - The matrix σ_k is invertible for any $k \in \mathbb{Z}/4\mathbb{Z}$. (With the above properties, this is equivalent to saying that the blocks along the diagonal are invertible.)
  - The product of the σ_k is the identity: $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$.
- A morphism $\delta = (\delta_k)_{k\in\mathbb{Z}/4\mathbb{Z}} \in \mathrm{Hom}_{\mathfrak{S}\mathfrak{D}^\star}(\sigma, \widetilde{\sigma})$ between two objects $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ and $\widetilde{\sigma} = (\widetilde{\sigma}_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ is a family of four block matrices with the properties:
  - The block structure is given by the numbers r_j ($j \in \{1, \ldots, n\}$).
  - The matrix δ_k is block-diagonal for every $k \in \mathbb{Z}/4\mathbb{Z}$.
  - For any $k \in \mathbb{Z}/4\mathbb{Z}$, one has $\widetilde{\sigma}_k\delta_k = \delta_{k+1}\sigma_k$.

  Composition of morphisms is given by matrix multiplication.

Remark. Let us give an explanation of how one could think of objects and morphisms in the category of Stokes data $\mathfrak{S}\mathfrak{D}^\star$. This also gives an idea for making a link with the description of Stokes data in [38].

An object consists of four matrices which will correspond to the Stokes matrices describing the transition between the four sectors. We can therefore imagine them to be arranged in a "circle", i.e. a diagram of the form

$$\begin{tikzcd} & \bullet \arrow[dl, "\sigma_2"'] & \bullet \arrow[l, "\sigma_1"'] \\ \bullet \arrow[r, "\sigma_3"] & \bullet \arrow[ur, "\sigma_4"'] & \end{tikzcd}$$

One can think of the vertices $\bullet$ as vector spaces $\mathbf{k}^r = \bigoplus_{j=1}^n \mathbf{k}^{r_j}$ which (by the given grading) have two natural filtrations: The filtration $F_m\mathbf{k}^r = \bigoplus_{j=1}^m \mathbf{k}^{r_j}$ is respected by the matrices σ_1 and σ_3, whereas the filtration $F'_m\mathbf{k}^r = \bigoplus_{j=n-m+1}^n \mathbf{k}^{r_j}$ is respected by the matrices σ_2 and σ_4.

A morphism between two such diagrams can then be visualized as

$$\begin{tikzcd} & \bullet \arrow[dl, "\sigma_2"'] \arrow[dd, "\delta_2"] & \bullet \arrow[l, "\sigma_1"'] \arrow[dd, "\delta_1"] \\ \bullet \arrow[r, "\sigma_3"] \arrow[dd, "\delta_3"] & \bullet \arrow[ur, "\sigma_4"'] \arrow[dd, "\delta_4"] & \\ & \bullet \arrow[dl, "\widetilde{\sigma}_2"'] & \bullet \arrow[l, "\widetilde{\sigma}_1"'] \\ \bullet \arrow[r, "\widetilde{\sigma}_3"] & \bullet \arrow[ur, "\widetilde{\sigma}_4"'] & \end{tikzcd}$$

and the relations required in Definition 2.18 amount to saying that this diagram is commutative. The matrices δ_k respect the grading $\mathbf{k}^r = \bigoplus_{j=1}^n \mathbf{k}^{r_j}$, i.e. they are compatible with both filtrations considered above.

An intuitive reason why the σ_k are block-triangular, while the δ_k need to be block-diagonal is the following: The matrices σ_k are the transition matrices, which means that they describe isomorphisms on the boundaries of the sectors, where one has a well-defined ordering of the parameters $c_{(j)}$ (cf. Proposition 2.9). In contrast, the δ_k are meant to describe morphisms on the sectors S_k, where no pair of parameters has a global well-defined order. Therefore, δ_k must be compatible with any order of the $c_{(j)}$.

Proposition 2.19. *The functor*

$$\mathfrak{S}\mathfrak{D}^\star \to \mathrm{E}^\star_{\mathrm{Gauß}}(\mathrm{I}\mathbf{k}_{\mathbb{P}}), \sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}} \mapsto \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_\sigma$$

is an equivalence of categories, where $\mathcal{F}_\sigma$ is the sheaf glued from $\pi^{-1}\mathbf{k}_{S_k} \otimes \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$ via the matrices σ_k as in Section 2.4.

Proof. The functor is well-defined, and it acts on morphisms in the obvious way: a block-diagonal matrix δ_k represents an endomorphism of $\pi^{-1}\mathbf{k}_{S_k} \otimes \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$, and a compatible family $(\delta_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ can be uniquely glued to a morphism of sheaves $\mathcal{F}_\sigma \to \mathcal{F}_{\widetilde{\sigma}}$. This induces a morphism $\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_\sigma \to \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}_{\widetilde{\sigma}}$.

The functor is fully faithful since we can describe an inverse to the map

$$\mathrm{Hom}_{\mathfrak{S}\mathfrak{D}^\star}(\sigma,\widetilde{\sigma}) \longrightarrow \mathrm{Hom}_{\mathrm{E}^\star_{\text{Gauß}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})}(\mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathcal{F}_\sigma, \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathcal{F}_{\widetilde{\sigma}})$$

as follows: Tensoring a morphism $\mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathcal{F}_\sigma \to \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathcal{F}_{\widetilde{\sigma}}$ with $\pi^{-1}\mathbf{k}_{S_k}$ yields an endomorphism of $\pi^{-1}\mathbf{k}_{S_k} \otimes \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \bigoplus_{c\in C}\big(\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{C}}\big)^{r_c}$, which is given by a block diagonal matrix δ_k (cf. Proposition 2.9). Compatibility with the σ_k and $\widetilde{\sigma}_k$ is clear.

By Proposition 2.17, any object of $\mathrm{E}^\star_{\text{Gauß}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})$ is of the form $H \simeq \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}(\mathcal{M})$ for an object $\mathcal{M} \in \mathrm{Mod}^\star_{\text{Gauß}}(\mathcal{D}_{\mathbb{P}})$. Hence, by Theorem 2.15, there exists a family $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ such that $H \simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathcal{F}_\sigma$, which proves essential surjectivity. □

Again, it is possible to define a category of Stokes data and obtain a result analogous to Proposition 2.19 without fixing the ranks r_c.

The upshot of this section is the following corollary, which one could call a Riemann–Hilbert correspondence for D-modules of pure Gaussian type with fixed exponential factors, including ranks, and a fixed generic direction. Note that the right-hand side only involves objects from linear algebra, which determine the enhanced solutions of a D-module of pure Gaussian type.

Corollary 2.20. *There is an equivalence of categories* $\mathrm{Mod}^\star_{\text{Gauß}}(\mathcal{D}_{\mathbb{P}}) \xrightarrow{\sim} \mathfrak{S}\mathfrak{D}^\star$.

The corresponding functor assigns to a D-module of pure Gaussian type $\mathcal{M}$ its Stokes matrices. Since the category $\mathfrak{S}\mathfrak{D}^\star$ has isomorphisms which are not the identity, we observe that the Stokes matrices are not uniquely determined by $\mathcal{M}$. They depend – as we have already noted before – on a choice of the isomorphisms (2.5). This means that they are only unique up to conjugation by block-diagonal matrices in the following way: Two families $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ and $(\widetilde{\sigma}_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ of Stokes matrices are isomorphic if and only if there exist invertible block-diagonal matrices $(\delta_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ such that $\widetilde{\sigma}_k = \delta_{k+1}\sigma_k\delta_k^{-1}$.

Chapter 3.

Fourier–Laplace transform

This chapter is devoted to the study of the Fourier–Laplace transform of a D-module of pure Gaussian type. More precisely, we want to give – under appropriate assumptions – a transformation rule for the Stokes data associated to such a module. The computations will be carried out "topologically", meaning that we use the Riemann–Hilbert correspondence in order to translate the problem from the category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ into the category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}})$, which was the context for defining the Stokes multipliers in the preceding chapter. It will turn out that the computations to be carried out in the latter category reduce to computations in the category $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$ of sheaves of vector spaces on $\mathbb{C}\times\mathbb{R}$, or enhanced sheaves on $\mathbb{C}$ (cf. Section 1.2.2).

It can be deduced from theorems like the stationary phase formula (see [36, Theorem 5.1]) that the Fourier–Laplace transform of a D-module of pure Gaussian type is a again of pure Gaussian type ([38, Lemma 1.4])). This is why we can attach Stokes data of the same type to the Fourier–Laplace transform and hence the idea of a "transformation rule" mentioned above is sensible. We will, however, not use this knowledge for our considerations. It will instead follow a posterori as a corollary of our result.

The chapter is structured as follows: First, we recall the notion of Fourier–Laplace transform for D-modules and its counterpart in the category of enhanced ind-sheaves. Then, we perform computations in three cases with increasing complexity: Section 3.2 studies an exponential D-module of (pure) Gaussian type. In Section 3.3, we compute the Fourier–Laplace transform for enhanced sheaves of pure Gaussian type in the case of parameters with a common argument, and we thus recover a result of [38]. In the last section, we show that our methods apply also to more general cases.

3.1. Analytic and topological Fourier–Laplace transform

As before, let $\mathbb{P} = \mathbb{P}^1(\mathbb{C})$ be the (analytic) complex projective line. Denote by $\mathbb{A}^1$ the (algebraic) complex affine line.

Classically, for a module M over the Weyl algebra $\mathbb{C}[z]\langle\partial_z\rangle$, the Fourier–Laplace transform[5] $\widehat{M}$ is the $\mathbb{C}[w]\langle\partial_w\rangle$-module defined as follows: As a set, we have $\widehat{M} = M$, and the structure of a $\mathbb{C}[w]\langle\partial_w\rangle$-module is defined by $w \cdot m := \partial_z m$ and $\partial_w m := -z \cdot m$. It is well-known (see e.g. [16, p. 87]) that holonomic modules over the Weyl algebra correspond

[5] In the context of D-modules, the Fourier–Laplace transform is often just called Fourier transform or Laplace transform by other authors.

to holonomic algebraic $\mathcal{D}_{\mathbb{A}^1}$-modules. It was shown by N. Katz and G. Laumon (see [27, Lemme (7.1.4)]) that in the latter category the Fourier–Laplace transform can be expressed as an integral transform whose kernel is the algebraic D-module associated to the function e^{-zw}. Since (by GAGA, see [29, p. 75]) holonomic algebraic $\mathcal{D}_{\mathbb{A}^1}$-modules are in one-to-one correspondence with holonomic analytic $\mathcal{D}_{\mathbb{P}}$-modules $\mathcal{M}$ satisfying $\mathcal{M}(*\infty) \simeq \mathcal{M}$, it also makes sense to consider the Fourier–Laplace transform as a functor on analytic modules.

Consider the projections

$$\begin{tikzcd} & \mathbb{P}_z \times \mathbb{P}_w \arrow[dl, "p_z"'] \arrow[dr, "p_w"] & \\ \mathbb{P}_z & & \mathbb{P}_w \end{tikzcd}$$

where $\mathbb{P}_z$ denotes the complex projective line with affine coordinate z in the chart $\mathbb{C}_z \subset \mathbb{P}_z$ at 0, and similarly for $\mathbb{P}_w$. We denote by ∞_z and ∞_w the unique points outside the charts at 0. Note that $-zw$ represents a meromorphic function on $\mathbb{P}_z \times \mathbb{P}_w$ with poles at $(\{\infty_z\} \times \mathbb{P}_w) \cup (\mathbb{P}_z \times \{\infty_w\})$.

Definition 3.1. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}_z})$. We define the *Fourier–Laplace transform* ${}^{\mathsf{L}}\mathcal{M}$ of $\mathcal{M}$ by

$${}^{\mathsf{L}}\mathcal{M} := \mathsf{D}p_{w*}\big(\mathcal{E}^{-zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \otimes^{\mathsf{D}} \mathsf{D}p_z^*\mathcal{M}\big) \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}_w}).$$

This defines a functor ${}^{\mathsf{L}}(\bullet)\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}_z}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}_w})$.

There is a similar transform for enhanced ind-sheaves, which was introduced and studied in [26]. Note, however, that our definition involves an additional shift (as in [7]).

Definition 3.2. Let $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}_z})$. We define the *enhanced Fourier–Sato transform* (or the *topological Fourier–Laplace transform*) ${}^{\mathsf{L}}H$ of H by

$${}^{\mathsf{L}}H := \mathrm{E}p_{w!!}\big(\mathbb{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_z^{-1}H\big)[1] \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}_w}).$$

This defines a functor ${}^{\mathsf{L}}(\bullet)\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}_z}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}_w})$.

Remark. The definition of the enhanced Fourier–Sato transform differs between various authors. In [26] and [4], it is stated without a shift, which then forces a shift in the statements below. In [7], the authors use the enhanced sheaf $\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w}$ instead of the enhanced ind-sheaf $\mathbb{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w}$. This does not make a difference as long as we apply the enhanced Fourier–Sato transform to stable objects $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}_z})$ (i.e. objects with $\mathbf{k}^{\mathrm{E}}_{\mathbb{P}_z} \overset{+}{\otimes} H \simeq H$) because

$$\begin{aligned}
\mathbb{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_z^{-1}H &\simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}_z\times\mathbb{P}_w} \overset{+}{\otimes} \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_z^{-1}H \\
&\simeq \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_z^{-1}\big(\mathbf{k}^{\mathrm{E}}_{\mathbb{P}_z} \overset{+}{\otimes} H\big) \\
&\simeq \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_z^{-1}H.
\end{aligned}$$

Note that $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$ is stable for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, since $\mathbf{k}_X^{\mathrm{E}} \overset{+}{\otimes} \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{O}_X) \overset{+}{\otimes} \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{O}_X \otimes^{\mathsf{D}} \mathcal{M}) \simeq \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$.

An important observation on our way to describing the Fourier–Laplace transform of a D-module of pure Gaussian type is the compatibility of these two transformations with the enhanced solution functor (cf. [26, Theorem 4.17]) and the stabilization functor. We define the projections

$$\begin{tikzcd} & \mathbb{C}_z \times \mathbb{C}_w \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{C}_z & & \mathbb{C}_w \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \arrow[dl, "\widetilde{p}"'] \arrow[dr, "\widetilde{q}"] & \\ \mathbb{C}_z \times \mathbb{R} & & \mathbb{C}_w \times \mathbb{R} \end{tikzcd}$$

where $\widetilde{p} = p \times \mathrm{id}_{\mathbb{R}}$ and $\widetilde{q} = q \times \mathrm{id}_{\mathbb{R}}$. Note that p is induced by p_z and q is induced by p_w.

Lemma 3.3. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}_z})$ be of pure Gaussian type and $\mathcal{S}ol_{\mathbb{P}_z}^{\mathrm{E}}(\mathcal{M}) \simeq \mathbf{k}_{\mathbb{P}_z}^{\mathrm{E}} \overset{+}{\otimes} \mathcal{F}$ with $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}_z \times \mathbb{R}})$. One has isomorphisms in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{P}_w})$*

$$\mathcal{S}ol_{\mathbb{P}_w}^{\mathrm{E}}({}^{\mathsf{L}}\mathcal{M}) \simeq {}^{\mathsf{L}}\mathcal{S}ol_{\mathbb{P}_z}^{\mathrm{E}}(\mathcal{M}) \simeq \mathbf{k}_{\mathbb{P}_w}^{\mathrm{E}} \overset{+}{\otimes} {}^{\mathsf{L}}\mathcal{F} \simeq \mathbf{k}_{\mathbb{P}_w}^{\mathrm{E}} \overset{+}{\otimes} \mathrm{R}\widetilde{q}_!\big(\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z \times \mathbb{C}_w | \mathbb{C}_z \times \mathbb{C}_w} \overset{*}{\otimes} \widetilde{p}^{-1}\mathcal{F}\big)[1].$$

Proof. The proof of the first isomorphism simply applies the functorialities of $\mathcal{S}ol_{\mathbb{P}}^{\mathrm{E}}$ (see Proposition 1.11) and (1.5). The only thing one has to pay attention to is that the formula for the direct image holds since a D-module of pure Gaussian type is "algebraic", i.e. comes from an algebraic (holonomic) D-module via analytification and hence admits a good filtration (cf. [16, Theorem 2.1.3]).

The second isomorphism is easily proved using [6, Proposition 4.7.14] together with commutativity and associativity of convolution ([6, Lemma 4.1.4, Lemma 4.1.5]).

For the last isomorphism, we first remark that ${}^{\mathsf{L}}\mathcal{F}$ denotes the enhanced Fourier–Sato transform of the enhanced ind-sheaf associated to $\mathcal{F}$ (see Section 1.2.2). We recall the following facts from the theory of enhanced ind-sheaves in [6]:

For a morphism $f\colon X \to Y$ of complex manifolds, the functor $\mathrm{E}f^{-1}\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_Y) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$ is, by definition, given by the functor $f_{\mathbb{R}_\infty}^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{Y \times \mathbb{R}_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X \times \mathbb{R}_\infty})$. (The notation $f_{\mathbb{R}_\infty}$ is taken from [8] and this morphism was denoted by $\tilde{f}$ in [6].) For $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{Y \times \mathbb{R}})$, there is an isomorphism $f_{\mathbb{R}_\infty}^{-1}\mathcal{G} \simeq f_{\mathbb{R}}^{-1}\mathcal{G}$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{X \times \mathbb{R}_\infty})$, where $f_{\mathbb{R}} = f \times \mathrm{id}_{\mathbb{R}}$ (see [6, Remark 3.3.21]). Similarly, for $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X \times \mathbb{R}})$, the proper direct image is given by $\mathrm{E}f_{!!}\mathcal{G} \simeq \mathrm{R}f_{\mathbb{R}!}\mathcal{G}$ if f is proper. Applying this, we get

$$\begin{aligned} {}^{\mathsf{L}}\mathcal{F} &\simeq \mathrm{E}p_{w!!}\big(\mathbb{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z \times \mathbb{C}_w | \mathbb{P}_z \times \mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_z^{-1}\mathcal{F}\big)[1] \\ &\simeq \mathrm{E}p_{w!!}\big(\mathbf{k}^{\mathrm{E}}_{\mathbb{P}_z \times \mathbb{P}_w} \overset{+}{\otimes} \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z \times \mathbb{C}_w | \mathbb{P}_z \times \mathbb{P}_w} \overset{+}{\otimes} (p_z)_{\mathbb{R}}^{-1}\mathcal{F}\big)[1] \\ &\simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}_w} \overset{+}{\otimes} \mathrm{E}p_{w!!}\big(\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z \times \mathbb{C}_w | \mathbb{P}_z \times \mathbb{P}_w} \overset{*}{\otimes} (p_z)_{\mathbb{R}}^{-1}\mathcal{F}\big)[1] \\ &\simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}_w} \overset{+}{\otimes} \mathrm{R}(p_w)_{\mathbb{R}!}\big(\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z \times \mathbb{C}_w | \mathbb{P}_z \times \mathbb{P}_w} \overset{*}{\otimes} (p_z)_{\mathbb{R}}^{-1}\mathcal{F}\big)[1]. \end{aligned}$$

In the third isomorphism, we have used [6, Proposition 4.7.14] and the compatibility of $\overset{*}{\otimes}$ and $\overset{+}{\otimes}$.

Since $\mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \overset{+}{\otimes} \mathbf{k}_{\mathbb{P}}^{\mathrm{E}} \simeq \mathbf{k}_{\mathbb{P}}^{\mathrm{E}}$, the last step is to reduce the inverse and direct image operations along p_z and p_w to those along $\widetilde{p}$ and $\widetilde{q}$ on the level of enhanced sheaves. For this, consider the commutative diagram

$$\begin{tikzcd} \mathbb{P}_z \times \mathbb{P}_w \times \mathbb{R} \arrow[r, "(p_z)_{\mathbb{R}}"] & \mathbb{P}_z \times \mathbb{R} \\ \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \arrow[u, hook, "\hat{\imath}"] \arrow[r, "\widetilde{p}"] & \mathbb{C}_z \times \mathbb{R}, \arrow[u, hook, "\tilde{\imath}_z"'] \end{tikzcd} \tag{3.1}$$

where the vertical arrows are inclusions. Now note that the enhanced sheaf $\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w}$ is supported in $\mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R}$ and hence is isomorphic to $\hat{\imath}_! \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \simeq \pi^{-1}\mathbf{k}_{\mathbb{C}_z\times\mathbb{C}_w} \otimes \hat{\imath}_! \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w}$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}_z\times\mathbb{P}_w\times\mathbb{R}})$. We get a chain of isomorphisms (emphasizing extension by zero in most places)

$$\begin{aligned} \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{P}_z\times\mathbb{P}_w} \overset{*}{\otimes} (p_z)_{\mathbb{R}}^{-1}\mathcal{F} &\simeq \left(\pi^{-1}\mathbf{k}_{\mathbb{C}_z\times\mathbb{C}_w} \otimes \hat{\imath}_! \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w}\right) \overset{*}{\otimes} (p_z)_{\mathbb{R}}^{-1}\, \tilde{\imath}_{z!}\mathcal{F} \\ &\simeq \hat{\imath}_! \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \overset{*}{\otimes} \left(\pi^{-1}\mathbf{k}_{\mathbb{C}_z\times\mathbb{C}_w} \otimes (p_z)_{\mathbb{R}}^{-1}\, \tilde{\imath}_{z!}\mathcal{F}\right) \\ &\simeq \hat{\imath}_! \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \overset{*}{\otimes} \hat{\imath}_!\, \hat{\imath}^{-1} (p_z)_{\mathbb{R}}^{-1}\, \tilde{\imath}_{z!}\mathcal{F} \\ &\simeq \hat{\imath}_! \mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \overset{*}{\otimes} \hat{\imath}_!\, \widetilde{p}^{-1} \tilde{\imath}^{-1} \tilde{\imath}_{z!}\mathcal{F} \\ &\simeq \hat{\imath}_!\left(\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \overset{*}{\otimes} \widetilde{p}^{-1}\mathcal{F}\right). \end{aligned}$$

The second isomorphism follows from a statement analogous to (1.4) for enhanced sheaves. The fourth isomorphism uses the commutativity of the square (3.1). The last step uses the projection formula and the fact that one has $\hat{\imath}^{-1}\hat{\imath}_! \simeq \mathrm{id}$, and similarly for $\tilde{\imath}_z$ (cf. Lemma A.1).

Finally, consider the commutative square

$$\begin{tikzcd} \mathbb{P}_z \times \mathbb{P}_w \times \mathbb{R} \arrow[r, "(p_w)_{\mathbb{R}}"] & \mathbb{P}_w \times \mathbb{R} \\ \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \arrow[u, hook, "\hat{\imath}"] \arrow[r, "\widetilde{q}"] & \mathbb{C}_w \times \mathbb{R} \arrow[u, hook, "\tilde{\imath}_w"'] \end{tikzcd}$$

showing that $\mathrm{R}(p_w)_{\mathbb{R}!}\hat{\imath}_! \simeq \tilde{\imath}_{w!}\mathrm{R}\widetilde{q}_!$ and thus completing the proof. □

The lemma just proved states that ${}^{\mathsf{L}}\mathcal{M}$ is determined by the object ${}^{\mathcal{L}}\mathcal{F}$ defined next.

Definition 3.4. Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}})$ be an enhanced sheaf. We define its *enhanced Fourier–Sato transform* (or *topological Fourier–Laplace transform*) ${}^{\mathcal{L}}\mathcal{F}$ by

$$ {}^{\mathcal{L}}\mathcal{F} := \mathrm{R}\widetilde{q}_!\big(\mathsf{E}^{-\operatorname{Re} zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \overset{*}{\otimes} \widetilde{p}^{-1}\mathcal{F}\big)[1] \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}}). $$

This defines a functor ${}^{\mathcal{L}}(\bullet)\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$.

Note that ${}^{\mathsf{L}}\mathcal{F} \simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}_w} \overset{+}{\otimes} {}^{\mathcal{L}}\mathcal{F}$ (where the left-hand side denotes the enhanced Fourier–Sato transform of the enhanced ind-sheaf associated to $\mathcal{F}$), which was shown in the proof of Lemma 3.3. Comparing with the remark above, the reason why there is no isomorphism ${}^{\mathsf{L}}\mathcal{F} \simeq {}^{\mathcal{L}}\mathcal{F}$ is that $\mathcal{F}$ is not stable as an object of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathbb{C}_z})$.

3.2. Warming up: A single exponential

The simplest example of a D-module of pure Gaussian type is $\mathcal{M} = \mathcal{E}^{-\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{P}}$ for some $c \in \mathbb{C}^\times$. In this case, no (nontrivial) Stokes phenomenon appears since the category of Stokes data for $C = \{c\}$, $r_c = 1$ and arbitrary θ_0 has only one isomorphism class. Note that this is also the case for nonzero r_c as long as C only consists of one element. The intuitive reason is that the Stokes phenomenon describes the interplay between *different* exponential factors (whose relation is incorporated in the upper and lower block-triangular structure of the Stokes matrices).

We take this example to perform a first topological computation of the Fourier–Laplace transform although the result could be obtained much faster applying, for example, the stationary phase formula (cf. [36]).

We know that

$$ \mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}\big(\mathcal{E}^{-\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{P}}\big) \simeq \mathbb{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{P}} \simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}|\mathbb{C}}, $$

so the following proposition is the key point in determining the Fourier–Laplace transform.

Proposition 3.5. *We have an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$ {}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}_z|\mathbb{C}_z} \simeq \mathsf{E}^{\operatorname{Re}\frac{1}{2c}w^2}_{\mathbb{C}_w|\mathbb{C}_w}. \tag{3.2} $$

Proof. Using the definition of the enhanced Fourier–Sato transform for enhanced sheaves and Lemma A.5, one calculates

$$ \begin{aligned} {}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathbb{C}_z|\mathbb{C}_z} &\simeq \mathrm{R}\widetilde{q}_!(\mathbf{k}_{\{t-\operatorname{Re} zw\geq 0\}} \overset{*}{\otimes} \widetilde{p}^{-1}\mathbf{k}_{\{t-\operatorname{Re}\frac{c}{2}z^2\geq 0\}})[1] \\ &\simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\operatorname{Re}(zw+\frac{c}{2}z^2)\geq 0\}}[1], \end{aligned} $$

where $\{t-\operatorname{Re}(zw+\frac{c}{2}z^2) \geq 0\} = \{(z,w,t) \in \mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R} \mid t-\operatorname{Re}(zw+\frac{c}{2}z^2) \geq 0\}$. Hence, it remains to show that the (derived) proper direct image $\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\operatorname{Re}(zw+\frac{c}{2}z^2)\}}$ is concentrated in degree one with cohomology sheaf $\mathbf{k}_{\{t+\operatorname{Re}\frac{1}{2c}w^2\geq 0\}}$.

We check this isomorphism on stalks first. This will give us some useful overview of the topological objects involved. Thereafter, we prove the desired isomorphism using the ideas arising from the local analysis.

The local picture: Stalks

Let $(\check{w}, \check{t}) \in \mathbb{C}_w \times \mathbb{R}$ be an arbitrary point. We get isomorphisms (cf. Lemma B.2)

$$\begin{aligned}\left(\mathrm{R}^l\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}}\right)_{(\check{w},\check{t})} &\simeq \mathrm{H}^l_\mathrm{c}\left(\widetilde{q}^{-1}(\check{w},\check{t}); \mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}}\big|_{\widetilde{q}^{-1}(\check{w},\check{t})}\right)\\ &\simeq \mathrm{H}^l_\mathrm{c}\left(\mathbb{C}_z\times\{\check{w}\}\times\{\check{t}\}; \mathbf{k}_{\{\check{t}-\mathrm{Re}(z\check{w}+\frac{c}{2}z^2)\geq 0\}\times\{\check{w}\}\times\{\check{t}\}}\right)\\ &\simeq \mathrm{H}^l_\mathrm{c}\left(\mathbb{C}_z; \mathbf{k}_{\{\check{t}-\mathrm{Re}(z\check{w}+\frac{c}{2}z^2)\geq 0\}}\right)\\ &\simeq \mathrm{H}^l_\mathrm{c}\left(\{\check{t}-\mathrm{Re}(z\check{w}+\frac{c}{2}z^2)\geq 0\}; \mathbf{k}\right).\end{aligned}$$

The aim is therefore to determine the compactly supported cohomology groups of the topological space $\Xi_{\check{w},\check{t}} := \{\check{t}-\mathrm{Re}(z\check{w}+\frac{c}{2}z^2)\geq 0\} \subseteq \mathbb{C}_z$ (equipped with the subspace topology induced by the Euclidean topology on $\mathbb{C}_z$).

We will write the occurring complex numbers with their real and imaginary parts as $c = c_1 + ic_2$, $z = z_1 + iz_2$ and $\check{w} = \check{w}_1 + i\check{w}_2$. We will denote by $|c|$ the absolute value of c, i.e. $|c|^2 = c_1^2 + c_2^2$. It is convenient to distinguish two cases.

Case $c_1 \neq 0$

In real coordinates, the inequality defining $\Xi_{\check{w},\check{t}}$ is written as

$$\check{t} - \frac{1}{2}\left(c_1z_1^2 - c_1z_2^2 - 2c_2z_1z_2\right) - z_1\check{w}_1 + z_2\check{w}_2 \geq 0.$$

After completing the square twice and applying the (homeomorphic) coordinate transformation given by $x_1 := z_1 - \frac{c_2}{c_1}z_2 + \frac{\check{w}_1}{c_1}$ and $x_2 := z_2 + \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2}$, one can see that this is equivalent to

$$\frac{c_1}{2}x_1^2 - \frac{|c|^2}{2c_1}x_2^2 \leq \check{t} + \frac{\check{w}_1^2}{2c_1} - \frac{(c_1\check{w}_2 - c_2\check{w}_1)^2}{2c_1|c|^2}. \tag{3.3}$$

Note that the right-hand side of (3.3) is nothing but $\check{t} + \mathrm{Re}\,\frac{1}{2c}\check{w}^2$, which gives a first hint at the connection with the right-hand side of (3.2). We will write for short $\kappa(\check{w},\check{t}) := \check{t} + \mathrm{Re}\,\frac{1}{2c}\check{w}^2$.

For $\kappa(\check{w},\check{t}) < 0$, inequality (3.3) becomes

$$\frac{c_1}{2\kappa(\check{w},\check{t})}x_1^2 - \frac{|c|^2}{2c_1\kappa(\check{w},\check{t})}x_2^2 \geq 1,$$

which describes the (closure of the) region outside the two branches of a hyperbola (see Fig. 3.1 (a)). Its compactly supported cohomology vanishes in all degrees (cf. Lemma B.1 (i)), i.e.

$$\mathrm{H}^l_\mathrm{c}(\Xi_{\check{w},\check{t}}; \mathbf{k}) \simeq 0 \quad \text{for all } l \in \mathbb{Z} \quad (\text{if } \kappa(\check{w},\check{t}) < 0).$$

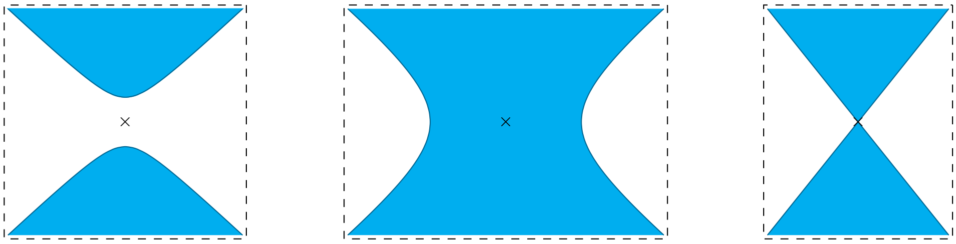
On the other hand, if $\kappa(\breve{w},\breve{t}) \geq 0$, inequality (3.3) describes the (closed) region between the two branches of a hyperbola (see Fig. 3.1 (b)): The inequality can be written in the form

$$\frac{c_1}{2\kappa(\breve{w},\breve{t})}x_1^2 - \frac{|c|^2}{2c_1\kappa(\breve{w},\breve{t})}x_2^2 \leq 1$$

if $\kappa(\breve{w},\breve{t}) > 0$. In the case where $\kappa(\breve{w},\breve{t}) = 0$, the hyperbola is degenerate (see Fig. 3.1 (iii)), but the region is still simply connected and can be treated in the same way as for positive $\kappa(\breve{w},\breve{t})$. The cohomology of this type of region is (cf. Lemma B.1 (ii))

$$\mathrm{H}^1_{\mathrm{c}}(\Xi_{\breve{w},\breve{t}};\mathbf{k}) \simeq \mathbf{k} \qquad \text{and} \qquad \mathrm{H}^l_{\mathrm{c}}(\Xi_{\breve{w},\breve{t}};\mathbf{k}) \simeq 0 \quad \text{for } l \neq 1 \quad (\text{if } \kappa(\breve{w},\breve{t}) \geq 0).$$

Altogether, we have seen that the stalks of $\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}}$ and $\mathbf{k}_{\{t+\mathrm{Re}\,\frac{1}{2c}w^2\geq 0\}}[-1]$ are the same in each degree and at each point of $\mathbb{C}_w \times \mathbb{R}$. In particular, both are concentrated in one degree.



(a) $\kappa(\breve{w},\breve{t}) < 0$ (b) $\kappa(\breve{w},\breve{t}) > 0$ (c) $\kappa(\breve{w},\breve{t}) = 0$

Figure 3.1.: The hyperbolic regions $\Xi_{\breve{w},\breve{t}} \subset \mathbb{R}^2$ (in standard form) in the case $c_1 \neq 0$. (The pictures sketch the situation for $c_1 > 0$. Orientations are different for $c_1 < 0$.)

Case $c_1 = 0$

If $c_1 = 0$, the inequality defining $\Xi_{\breve{w},\breve{t}}$ reduces to

$$\breve{t} + c_2 z_1 z_2 - z_1\breve{w}_1 + z_2\breve{w}_2 \geq 0.$$

After the transformation $x_1 := z_1 + \frac{\breve{w}_2}{c_2}$ and $x_2 := z_2 - \frac{\breve{w}_1}{c_2}$, this becomes

$$c_2 x_1 x_2 \geq -\Big(\breve{t} + \frac{\breve{w}_1\breve{w}_2}{c_2}\Big).$$

(Note that $c_2 \neq 0$ in this case since $c \in \mathbb{C}^\times$.) In order to write the hyperbola in standard form also in this case, one applies another coordinate transform given by $y_1 = \frac{1}{2}(-x_1+x_2)$, $y_2 = \frac{1}{2}(x_1+x_2)$ to obtain

$$c_2 y_1^2 - c_2 y_2^2 \leq \breve{t} + \frac{\breve{w}_1\breve{w}_2}{c_2}.$$

Now everything is analogous to the previous case: We write for short $\kappa(\breve{w},\breve{t}) := \breve{t} + \frac{\breve{w}_1\breve{w}_2}{c_2}$

and observe that $\kappa(\check{w},\check{t}) = \check{t} + \mathrm{Re}\,\frac{1}{2c}\check{w}^2$. For $\kappa(\check{w},\check{t}) < 0$ (resp. $\kappa(\check{w},\check{t}) \geq 0$), the region described by this inequality is of the form shown in Fig. 3.1 (a) (resp. (b) and (c)), so these cases are homeomorphic to the ones we have seen before.

The global picture: Isomorphism of sheaves

We now want to prove that we have an isomorphism

$$\mathrm{R}\widetilde{q}_! \mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}} \simeq \mathbf{k}_{\{t+\mathrm{Re}\,\frac{1}{2c}w^2 \geq 0\}}[-1].$$

The idea is to replace $\{t - \mathrm{Re}(zw + \frac{c}{2}z^2) \geq 0\}$ (whose cross-sections for fixed w and t are regions bounded by hyperbolas) by a set A whose cross-section for fixed w and t is a generator (Borel–Moore cycle) of the the compactly supported cohomology of the corresponding region.

In the case $c_1 \neq 0$, for fixed w, t with $t + \mathrm{Re}\,\frac{1}{2c}w^2 \geq 0$, such a generator is given by the line $z_1 - \frac{c_2}{c_1}z_2 + \frac{w_1}{c_1} = 0$ (which corresponds to the vertical axis $x_1 = 0$ in Fig. 3.1 (b) and (c)). We therefore consider the set

$$A := \left\{(z,w,t) \in \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \,\middle|\, t + \mathrm{Re}\,\frac{1}{2c}w^2 \geq 0, z_1 - \frac{c_2}{c_1}z_2 + \frac{w_1}{c_1} = 0\right\}.$$

By definition, A is a closed subset of $\{t - \mathrm{Re}(zw + \frac{c}{2}z^2) \geq 0\}$, so we have an exact sequence in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}})$

$$0 \longrightarrow \mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}\smallsetminus A} \longrightarrow \mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}} \longrightarrow \mathbf{k}_A \longrightarrow 0.$$

If we apply the functor $\mathrm{R}\widetilde{q}_!$ to this short exact sequence of sheaves, we obtain a distinguished triangle

$$\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}\smallsetminus A} \longrightarrow \mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}} \longrightarrow \mathrm{R}\widetilde{q}_!\mathbf{k}_A \xrightarrow{+1}.$$

Now, note that $\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}\smallsetminus A} \simeq 0$ since we can calculate the stalk at any point $(\check{w},\check{t})$ and in any degree as in the prequel, and these stalks are compactly supported cohomology groups of spaces as in Fig. 3.1 (a) or Fig. 3.2 (cf. Lemma B.1 (iii)). Hence, $\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}} \simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_A$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \arrow[rrrr, bend left, "\widetilde{q}"] & A \arrow[l, hook', "j_A"'] \arrow[r, "a"] \arrow[d, "g"] & \{t + \mathrm{Re}\,\frac{1}{2c}w^2 \geq 0\} \arrow[r, hook, "j"] \arrow[d, "\rho"] & \mathbb{C}_w \times \mathbb{R} \\
& \mathbb{R} \arrow[r, "\rho_{\mathbb{R}}"] & \{\mathrm{pt}\}. &
\end{tikzcd}$$

Here, the map a is given by projection (so it is the restriction of $\widetilde{q}$). Furthermore, the

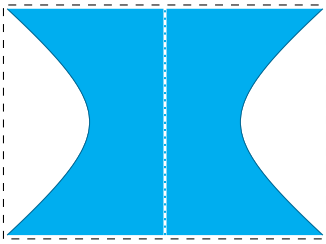


Figure 3.2.: The stalks of the cohomology sheaves of $\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}\smallsetminus A}$ at points $(\check{w},\check{t})$ with $\kappa(\check{w},\check{t})\geq 0$ are compactly supported cohomology groups of spaces of this form: The region from Fig. 3.1 (b) or (c) minus the vertical axis of the hyperbola.

map g is given by $(z,w,t)\mapsto \operatorname{Im} z$, and it is easy to check that the square is Cartesian. In particular, $\{t+\mathrm{Re}\,\frac{1}{2c}w^2\geq 0\}\times\mathbb{R}\to A$, $((w,t),y)\mapsto(\frac{c_2}{c_1}y-\frac{w_1}{c_1}+iy,w,t)$ is a homeomorphism (i.e. the Borel–Moore cycle is parametrized by y). Using this diagram, it follows that

$$\begin{aligned}\mathrm{R}\widetilde{q}_!\mathbf{k}_{\{t-\mathrm{Re}(zw+\frac{c}{2}z^2)\geq 0\}} &\simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_A \simeq \mathrm{R}\widetilde{q}_! j_{A!}\mathbf{k}_A \simeq j_!\mathrm{R}a_!\mathbf{k}_A \simeq j_!\mathrm{R}a_! g^{-1}\mathbf{k}_{\mathbb{R}}\\ &\simeq j_!\rho^{-1}\mathrm{R}\rho_{\mathbb{R}!}\mathbf{k}_{\mathbb{R}} \overset{(*)}{\simeq} j_!\rho^{-1}\mathbf{k}[-1]\simeq j_!\mathbf{k}_{\{t+\mathrm{Re}\,\frac{1}{2c}w^2\geq 0\}}[-1]\\ &\simeq \mathbf{k}_{\{t+\mathrm{Re}\,\frac{1}{2c}w^2\geq 0\}}[-1],\end{aligned}$$

which is what we wanted. The isomorphism $(*)$ follows because the compactly supported cohomology of the real line is concentrated (and one-dimensional) in degree one (which also follows from Lemma B.1 (ii), for example). The case $c_1=0$ is analogous. $\square$

We immediately get the following result.

Corollary 3.6. *The Fourier–Laplace transform of* $\mathcal{E}_{\mathbb{C}|\mathbb{P}}^{-\frac{c}{2}z^2}$ *for* $c\in\mathbb{C}^\times$ *is given by*

$$^{\mathsf{L}}\mathcal{E}_{\mathbb{C}_z|\mathbb{P}_z}^{-\frac{c}{2}z^2}\simeq\mathcal{E}_{\mathbb{C}_w|\mathbb{P}_w}^{\frac{1}{2c}w^2}.$$

Proof. By Lemma 3.3 and Proposition 3.5, we have

$$\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}_w}\big({}^{\mathsf{L}}\mathcal{E}_{\mathbb{C}_z|\mathbb{P}_z}^{-\frac{c}{2}z^2}\big)\simeq \mathsf{k}^{\mathrm{E}}_{\mathbb{P}_w}\overset{+}{\otimes}\mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\mathbb{C}_w|\mathbb{C}_w}\simeq\mathbb{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\mathbb{C}_w|\mathbb{P}_w}\simeq\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}_w}\big(\mathcal{E}^{\frac{1}{2c}w^2}_{\mathbb{C}_w|\mathbb{P}_w}\big),$$

where the last step uses (1.5), and the desired isomorphism follows from the full faithfulness of $\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}_w}$. $\square$

Thus, we have computed the Fourier–Laplace transform of a single exponential solely by means of topological methods (i.e. working with constructible sheaves and cohomology). The connection with the D-module side of the Riemann–Hilbert correspondence is made in the very first and last steps of our argumentation.

3.3. Sabbah's case: Aligned parameters

The next case we will treat is that of a D-module of pure Gaussian type C with $\arg c = \arg d$ for any $c, d \in C$, i.e. the parameter set C is "aligned" along a half-line through the origin. This is also the situation in which C. Sabbah formulated a result in [38]. Let us denote by $\arg C$ the common argument of all the parameters.

3.3.1. Main statement

Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be a D-module of pure Gaussian type C, where all the elements of C have the same argument $\arg C$.

Let us first determine the Stokes directions: If $c, d \in C$, we have $d = \lambda c$ for some $\lambda \in \mathbb{R}_{>0}$. The union of the four Stokes lines is the set of solutions of

$$\operatorname{Re} cz^2 = \operatorname{Re} \lambda cz^2.$$

Writing $c = |c|e^{i \arg C}$ and $z = |z|e^{i \arg z}$ in polar coordinates and simplifying, this is equivalent to

$$|z|^2 \cos(\arg C + 2 \arg z) = 0,$$

and therefore the directions of the Stokes lines are

$$\frac{\pi}{4} - \frac{1}{2} \arg C + k\frac{\pi}{2}, \quad k \in \mathbb{Z}/4\mathbb{Z}.$$

The values are the same for any pair $c, d \in C$, so there are only four Stokes directions in total. In particular, we can choose $\theta_0 := -\frac{1}{2} \arg C$ as a generic direction. Note that these considerations involve a choice of $\frac{1}{2} \arg C$, and, where it is necessary to specify this choice, we will choose $\arg C \in (-\pi, \pi]$, which forces $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2})$.

It is known from Theorem 2.15 that

$$\mathcal{S}ol^{\mathrm{E}}_{\mathbb{P}}(\mathcal{M}) \simeq \mathbf{k}^{\mathrm{E}}_{\mathbb{P}} \overset{+}{\otimes} \mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma}.$$

Therefore, the main step in computing the Fourier–Laplace transform of $\mathcal{M}$ topologically is the proof of the following statement.

Theorem 3.7. *Let $C \subset \mathbb{C}^\times$ be a finite subset such that $\arg c = \arg d$ for any $c, d \in C$. Denote by $\arg C$ the common argument of the elements of C, and set $\theta_0 := -\frac{1}{2} \arg C$. Choose a numbering of the elements of C such that $c_{(1)} <_{\theta_0} \ldots <_{\theta_0} c_{(n)}$. Let $r_c \in \mathbb{Z}_{>0}$ for any $c \in C$, and set $r := \sum_{c \in C} r_c$ as well as $\mathbb{r} := (r_c)_{c \in C}$. Let $\sigma = (\sigma_k)_{k \in \mathbb{Z}/4\mathbb{Z}}$ be a family of four block matrices $\sigma_k \in \mathbf{k}^{r \times r}$ whose block structure is given by the numbers $r_{c_{(1)}}, \ldots, r_{c_{(n)}}$ (i.e. the jth diagonal block has size $r_{c_{(j)}} \times r_{c_{(j)}}$) and such that σ_1 and σ_3 are upper block-triangular, σ_2 and σ_4 are lower block-triangular and $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$. We define $\widehat{C} := -1/C = \{-\frac{1}{c} \mid c \in C\}$, $\widehat{\theta}_0 := \pi - \theta_0$, $\widehat{r}_{\widehat{c}} := r_{-1/\widehat{c}}$ for any $\widehat{c} \in \widehat{C}$ and $\widehat{\mathbb{r}} := (\widehat{r}_{\widehat{c}})_{\widehat{c} \in \widehat{C}}$. Then there is an isomorphism*

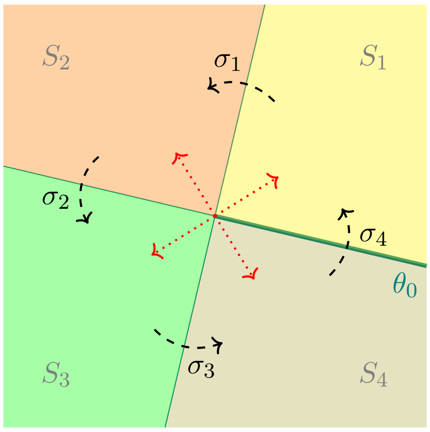
$$^{\mathcal{L}}\mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma} \simeq \mathcal{F}^{\widehat{C},\widehat{\theta}_0,\widehat{\mathbb{r}}}_{\sigma}.$$

In particular, the gluing matrices $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ remain the same (although sectors and exponential factors change).

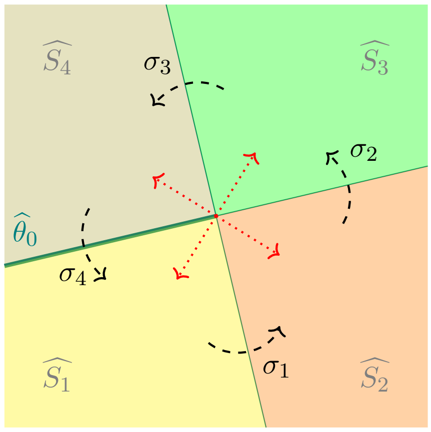
As a corollary, we get the following result, which was already obtained in the context of Stokes data attached to Stokes-filtered local systems by C. Sabbah (cf. [38, Lemma 1.4 and Theorem 4.2]). The statement is illustrated in Fig. 3.3. Recalling the fact that a D-module of pure Gaussian type is determined by its Stokes data (see Corollary 2.20), we have thus given a complete description of the Fourier–Laplace transform in this case.

Corollary 3.8. *Let $C \subset \mathbb{C}^\times$ be a finite subset whose elements have constant argument $\arg C$. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be of pure Gaussian type C and let $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ be Stokes multipliers for $\mathcal{M}$ with respect to the generic direction $\theta_0 = -\frac{1}{2}\arg C$. Then the Fourier–Laplace transform ${}^{\mathsf{L}}\mathcal{M}$ of $\mathcal{M}$ is of pure Gaussian type $\widehat{C} = -1/C$ and a family of Stokes multipliers for ${}^{\mathsf{L}}\mathcal{M}$ with respect to the generic direction $\widehat{\theta}_0 := \pi - \theta_0$ is given by $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$.*

Proof. It is easy to check that $\widehat{\theta}_0$ is indeed a generic direction for $\widehat{C}$. It is also not difficult to show that the ordering of the parameters carries over to the Fourier–Laplace transform, i.e. that $-\frac{1}{c} <_{\widehat{\theta}_0} -\frac{1}{d}$ whenever $c <_\theta d$ for $c, d \in C$. The fact that the ranks and Stokes multipliers remain unchanged follows from Theorem 3.7. $\square$



Original system



Fourier–Laplace transform

Figure 3.3.: The complex plane covered by four closed sectors, which are determined by the generic directions θ_0 and $\widehat{\theta}_0 = \pi - \theta_0$. (The red arrows indicate the Stokes directions.)
If a D-module of pure Gaussian type has a Hukuhara–Turrittin decomposition on each of the sectors S_k (on the left) with exponents $-\frac{c}{2}z^2$ and Stokes multipliers σ_k, then its Fourier–Laplace transform has a Hukuhara–Turrittin decomposition on the sectors $\widehat{S}_k$ (on the right) with exponents $\frac{1}{2c}w^2$ and Stokes multipliers $\widehat{\sigma}_k = \sigma_k$.

The rest of this section will be concerned with the proof of Theorem 3.7. The idea of the proof is as follows: In Section 3.3.2, we choose a suitable decomposition of the plane into four closed sectors $\mathcal{S}_k$, $k \in \mathbb{Z}/4\mathbb{Z}$, on which $\mathcal{F} = \mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}$ is trivialized as a direct sum of exponential enhanced sheaves. Setting $\mathcal{H}_+ := \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{H}_- := \mathcal{S}_3 \cup \mathcal{S}_4$, $\mathcal{S}_{k,k+1} := \mathcal{S}_k \cap \mathcal{S}_{k+1}$ and $L := \mathcal{S}_{41} \cup \mathcal{S}_{23}$, we can model the gluing of $\mathcal{F}$ from its restrictions to sectors in terms of short exact sequences in $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$:

$$\begin{gathered}
0 \longrightarrow \mathcal{F}_{\mathcal{H}_+} \longrightarrow \mathcal{F}_{\mathcal{S}_1} \oplus \mathcal{F}_{\mathcal{S}_2} \longrightarrow \mathcal{F}_{\mathcal{S}_{12}} \longrightarrow 0,\\
0 \longrightarrow \mathcal{F}_{\mathcal{H}_-} \longrightarrow \mathcal{F}_{\mathcal{S}_3} \oplus \mathcal{F}_{\mathcal{S}_4} \longrightarrow \mathcal{F}_{\mathcal{S}_{34}} \longrightarrow 0,\\
0 \longrightarrow \mathcal{F}_{L} \longrightarrow \mathcal{F}_{\mathcal{S}_{41}} \oplus \mathcal{F}_{\mathcal{S}_{23}} \longrightarrow \mathcal{F}_{\{0\}} \longrightarrow 0,\\
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{\mathcal{H}_+} \oplus \mathcal{F}_{\mathcal{H}_-} \longrightarrow \mathcal{F}_{L} \longrightarrow 0.
\end{gathered}$$

Using the isomorphims between $\mathcal{F}$ and a direct sum of exponential enhanced sheaves on sectors, we can make these sequences more explicit. Applying the enhanced Fourier–Sato transform, we obtain distinguished triangles (which will turn out to be just short exact sequences). As soon as we are able to describe the enhanced Fourier–Sato transform of exponential enhanced sheaves on sectors (which we will do in Section 3.3.3), we can determine step by step the enhanced Fourier–Sato transforms of $\mathcal{F}_{H_+}$, $\mathcal{F}_{H_-}$, $\mathcal{F}_L$, and finally of $\mathcal{F}$ in Section 3.3.4.

We will give a proof for the case where $\arg C \in (-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. $c_1 > 0$. The arguments for the other cases $c_1 < 0$ and $c_1 = 0$ work completely along the same lines. However, as in the proof of Proposition 3.5, we will choose coordinate transforms which are different for the cases $c_1 \neq 0$ and $c_1 = 0$. Moreover, the geometry of the hyperbolae depends on the sign of c_1 (or c_2 if $c_1 = 0$).

3.3.2. Sector decomposition

The first step is to choose sectors which will be convenient for our considerations. It will turn out that – when looking at stalks of exponential enhanced sheaves on sectors – we have to deal with geometric situations involving sectors and the hyperbolic regions $\Xi_{\check{w},\check{t}}$ from the proof of Proposition 3.5. In order to work with these hyperbolae, we prefer to write their equations in standard form, so we prefer the coordinates (x_1, x_2) to (z_1, z_2). Recall that we assume $c_1 > 0$. As seen in the proof of the proposition, the coordinate transform is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{c_2}{c_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{\check{w}_1}{c_1} \\ \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \end{pmatrix}. \tag{3.4}$$

We would like to find four closed sectors (with the origin as a common point) in the z-coordinates which transform to right-angled sectors (not necessarily centered at the origin) with horizontal and vertical boundaries in the x-coordinates. The properties of the hyperbola being in standard form and the sectors being of that form will make the situation easier.

The inverse coordinate transform is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{c_2}{c_1} \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \frac{\breve{w}_1}{c_1} \\ \frac{c_1 \breve{w}_2 - c_2 \breve{w}_1}{|c|^2} \end{pmatrix} \right),$$

and it is easy to see that the lines with directions $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$ through the point $\left(\frac{\breve{w}_1}{c_1}, \frac{c_1 \breve{w}_2 - c_2 \breve{w}_1}{|c|^2}\right)$ in x-coordinates correspond to the lines with directions $(1,0)^{\mathrm{T}}$ and $(\frac{c_2}{c_1}, 1)^{\mathrm{T}}$ through the origin in z-coordinates, respectively. Therefore, the sectors in z-coordinates should be chosen with these lines as borders, as shown in Fig. 3.4. The numbering is given by the generic direction θ_0: The first sector begins at the half-line which lies between the same Stokes directions as θ_0.

In symbols, the new sectors are (with $\arg C \in (-\frac{\pi}{2}, \frac{\pi}{2})$)

$$\begin{aligned}
\mathcal{S}_1 &:= \left\{ z \in \mathbb{C}_z \,\middle|\, \arg z \in \left[0, \frac{\pi}{2} - \arg C\right] \text{ if } z \neq 0 \right\}, \\
\mathcal{S}_2 &:= \left\{ z \in \mathbb{C}_z \,\middle|\, \arg z \in \left[\frac{\pi}{2} - \arg C, \pi\right] \text{ if } z \neq 0 \right\}, \\
\mathcal{S}_3 &:= \left\{ z \in \mathbb{C}_z \,\middle|\, \arg z \in \left[-\pi, -\frac{\pi}{2} - \arg C\right] \text{ if } z \neq 0 \right\}, \\
\mathcal{S}_4 &:= \left\{ z \in \mathbb{C}_z \,\middle|\, \arg z \in \left[-\frac{\pi}{2} - \arg C, 0\right] \text{ if } z \neq 0 \right\}.
\end{aligned}$$

Note that the origin is contained in all these sectors. As usual, we will denote the half-lines bounding the sectors by $\mathcal{S}_{k,k+1} := \mathcal{S}_k \cap \mathcal{S}_{k+1}$ for $k \in \mathbb{Z}/4\mathbb{Z}$.

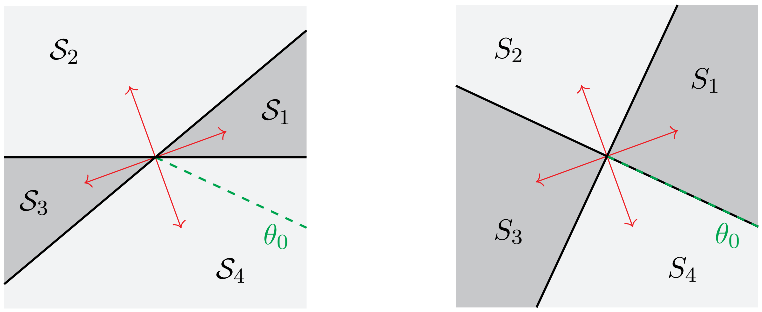


Figure 3.4.: The new sectors $\mathcal{S}_k$ (on the left) can be deformed into the standard sectors S_k without crossing a Stokes line. (The red arrows indicate the directions of the Stokes lines. The generic direction θ_0 is shown in green.)

It remains to justify the use of these sectors instead of the "standard sectors" given by $S_k = \{z \in \mathbb{C} \mid \arg z \in [\theta_0 + (k-1)\frac{\pi}{2}, \theta_0 + k\frac{\pi}{2}] \text{ if } z \neq 0\}$, which we have used up to now. This is done by the following lemma. It ensures that the Stokes multipliers remain the same if one moves the boundaries of the sectors without crossing a Stokes line.

Lemma 3.9. *Let S_1, S' and S_2 be three adjacent closed sectors at ∞ (of possibly infinite radius). Assume that their total angle is less than 2π (see Fig. 3.5) and that S' contains no Stokes direction.*

Let $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}})$ and assume that we have isomorphisms

$$\alpha_1\colon \mathcal{F}_{S_1\cup S'} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S_1\cup S'} \otimes \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}, \quad \alpha_2\colon \mathcal{F}_{S_2} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S_2} \otimes \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c},$$

whose restrictions to $S'\cap S_2$ are denoted by α_1' and α_2', respectively. Assume furthermore that the transition isomorphism $\alpha_2'\circ\alpha_1'^{-1}$ is given by a matrix σ_1.

Then there are isomorphisms

$$\widetilde{\alpha}_1\colon \mathcal{F}_{S_1} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S_1} \otimes \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}, \quad \widetilde{\alpha}_2\colon \mathcal{F}_{S_2\cup S'} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S_2\cup S'} \otimes \bigoplus_{c\in C} \big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$$

such that (denoting by $\widetilde{\alpha}_1'$ and $\widetilde{\alpha}_2'$ their restrictions to $S_1\cap S'$) the transition morphism $\widetilde{\alpha}_2'\circ\widetilde{\alpha}_1'^{-1}$ is given by the matrix σ_1.

As a consequence, Definition 2.14 defines the same sheaf $\mathcal{F}_\sigma^{C,\theta_0,\mathtt{r}}$ if we replace the sectors S_k by sectors (at ∞ of infinite radius) $\mathcal{S}_k$ such that $\mathcal{S}_k$ contains exactly the same Stokes directions as S_k for any $k\in\mathbb{Z}/4\mathbb{Z}$.

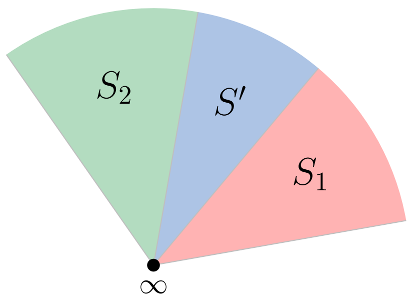


Figure 3.5.: The situation of Lemma 3.9: If S' contains no Stokes direction, trivializations of an enhanced sheaf of pure Gaussian type on $S_1\cup S'$ and S_2 induce trivializations on S_1 and $S_2\cup S'$ with the same transition matrix. In other words, S' can be considered a part of either of the two sectors without changing the rest of the gluing data.

Proof. Let us write for short $\mathsf{M} := \bigoplus_{c\in C}\big(\mathsf{E}_{\mathbb{C}|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}\big)^{r_c}$.

The morphism $\widetilde{\alpha}_1$ is obtained by applying the functor $\pi^{-1}\mathbf{k}_{S_1}\otimes(\bullet)$ to α_1. To construct $\widetilde{\alpha}_2$, we glue two morphisms via a short exact sequence: The morphism α_1 induces $\pi^{-1}\mathbf{k}_{S'}\otimes\alpha_1\colon \mathcal{F}_{S'} \xrightarrow{\sim} \pi^{-1}\mathbf{k}_{S'}\otimes\mathsf{M}$. Moreover, the matrix σ_1 represents an automorphism of $\pi^{-1}\mathbf{k}_{S'\cap S_2}\otimes\mathsf{M}$. Since there are no Stokes directions in S', it extends to an automorphism of $\pi^{-1}\mathbf{k}_{S'}\otimes\mathsf{M}$. Therefore, we obtain $\widetilde{\alpha}_2$ from the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{S_2\cup S'} \arrow[r] \arrow[d, dashed, "\widetilde{\alpha}_2"] & \mathcal{F}_{S_2} \oplus \mathcal{F}_{S'} \arrow[r] \arrow[d, "\alpha_2 \oplus \sigma_1\circ(\pi^{-1}\mathbf{k}_{S'}\otimes\alpha_1)"] & \mathcal{F}_{S'\cap S_2} \arrow[r] \arrow[d, "\pi^{-1}\mathbf{k}_{S'\cap S_2}\otimes\alpha_2"] & 0 \\
0 \arrow[r] & \pi^{-1}\mathbf{k}_{S_2\cup S'}\otimes\mathsf{M} \arrow[r] & \pi^{-1}\mathbf{k}_{S_2}\otimes\mathsf{M}\oplus\pi^{-1}\mathbf{k}_{S'}\otimes\mathsf{M} \arrow[r, "\mathrm{can}-\mathrm{can}"] & \pi^{-1}\mathbf{k}_{S'\cap S_2}\otimes\mathsf{M} \arrow[r] & 0
\end{tikzcd}$$

The third arrow in the lower row is the difference of the two canonical maps, which makes the square on the right commute and hence guarantees the existence of the blue vertical arrow on the left.

To find the new transition matrix on $S_1 \cap S'$, we tensor this diagram by $\pi^{-1}\mathbf{k}_{S_1\cap S'}$. This yields

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{S_1\cap S'} \arrow[r] \arrow[d, "\pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes\widetilde{\alpha}_2"] & \mathcal{F}_{\{0\}} \oplus \mathcal{F}_{S_1\cap S'} \arrow[r] \arrow[d, "\pi^{-1}\mathbf{k}_{\{0\}}\otimes\alpha_2 \,|\, \sigma_1\circ(\pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes\alpha_1)"] & \mathcal{F}_{\{0\}} \arrow[r] \arrow[d, "\pi^{-1}\mathbf{k}_{\{0\}}\otimes\alpha_2"] & 0 \\
0 \arrow[r] & \pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes \mathsf{M} \arrow[r] & \pi^{-1}\mathbf{k}_{\{0\}}\otimes \mathsf{M} \oplus \pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes \mathsf{M} \arrow[r, "\mathrm{can}-\mathrm{can}"] & \pi^{-1}\mathbf{k}_{\{0\}}\otimes \mathsf{M} \arrow[r] & 0.
\end{tikzcd}$$

(This is the diagram for the case in which the sectors have infinite radius, i.e. a common point at the origin. In the case of finite radius, $\{0\}$ is replaced by $\varnothing$, i.e. we have 0 in place of $\mathcal{F}_{\{0\}}$ etc.) It follows that $\pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes\widetilde{\alpha}_2 = \sigma_1\circ(\pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes\alpha_1)$. By construction, $\pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes\widetilde{\alpha}_1 = \pi^{-1}\mathbf{k}_{S_1\cap S'}\otimes\alpha_1$, so the transition map is given by σ_1 as desired. $\square$

3.3.3. Exponential enhanced sheaves on closed sectors

The aim is now to compute the enhanced Fourier–Sato transforms of $\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_k|\mathbb{C}_z}$, $\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{k,k+1}|\mathbb{C}_z}$ and $\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}|\mathbb{C}_z}$ for $k\in\mathbb{Z}/4\mathbb{Z}$. As mentioned, we assume $c = c_1 + ic_2 \in \mathbb{C}^\times$ with $c_1 > 0$.

In general, the enhanced Fourier–Sato transform of an exponential enhanced sheaf can be computed as

$$\begin{aligned}
{}^{\mathcal{L}}\mathsf{E}^{\varphi}_{W|\mathbb{C}_z} &= \mathrm{R}\widetilde{q}_! \left(\mathbf{k}_{\{t-\operatorname{Re} zw\geq 0\}} \overset{*}{\otimes} \widetilde{p}^{-1}\left(\pi^{-1}\mathbf{k}_W \otimes \mathbf{k}_{\{t+\varphi(z)\geq 0\}}\right)\right)[1] \\
&\simeq \mathrm{R}\widetilde{q}_! \left(\pi^{-1}\mathbf{k}_{W\times\mathbb{C}_w} \otimes \left(\mathbf{k}_{\{t-\operatorname{Re} zw\geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t+\varphi(z)\geq 0\}}\right)\right)[1] \\
&\simeq \mathrm{R}\widetilde{q}_! \mathbf{k}_{\{(z,w,t)\in\mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}\,|\, z\in W,\, t-\operatorname{Re} zw+\varphi(z)\geq 0\}}[1].
\end{aligned} \tag{3.5}$$

Here $W \subseteq \mathbb{C}_z$ is a locally closed subset, and $\varphi\colon \mathbb{C}_z \to \mathbb{R}$ is a continuous function. Note that the projection π is not the same in the first and second line: In the first line, we use $\pi\colon \mathbb{C}_z\times\mathbb{R}\to\mathbb{C}_z$, and in the second line we use $\pi\colon \mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}\to\mathbb{C}_z\times\mathbb{C}_w$. In the first isomorphism, we have used (1.4) for enhanced sheaves.

In particular, the stalks of the cohomology sheaves at a point $(\check{w},\check{t})\in\mathbb{C}_w\times\mathbb{R}$ are determined by the topology of the intersection of two subspaces of the plane $\mathbb{C}_z$, namely

$$\mathrm{H}^l\big({}^{\mathcal{L}}\mathsf{E}^{\varphi}_{W|\mathbb{C}_z}\big)_{(\check{w},\check{t})} \simeq \mathrm{H}^{l+1}_{\mathrm{c}}\big(W\cap\{z\in\mathbb{C}\mid \check{t}-\operatorname{Re} z\check{w}+\varphi(z)\geq 0\};\mathbf{k}\big). \tag{3.6}$$

Exponentials on the sectors $\mathcal{S}_k$

Let us start with the enhanced sheaf $\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_k|\mathbb{C}_z}$. We proceed analogously to the proof of Proposition 3.5.

To get an idea of what the result could be, we consider the stalks of the cohomology sheaves of ${}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_k|\mathbb{C}_z}$. By (3.6), they are given by the compactly supported cohomology groups of the intersection of the space $\Xi_{\check{w},\check{t}} = \{\check{t} - \operatorname{Re}(z\check{w} + \frac{c}{2}z^2) \geq 0\}$ with the sector $\mathcal{S}_k$. Since the other cases work analogously, we only consider the case $k = 1$ here.

Applying the coordinate transformation (3.4), which does not change compactly supported cohomology since the latter is invariant under homeomorphisms, the space to be considered is the intersection of the (hyperbolic) region given by

$$\frac{c_1}{2}x_1^2 - \frac{|c|^2}{2c_1}x_2^2 \leq \check{t} + \operatorname{Re}\frac{1}{2c}\check{w}^2$$

and the sector given by

$$x_1 \geq \frac{\check{w}_1}{c_1}, \quad x_2 \geq \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2}.$$

When studying the topology of this intersection, one finds that it never has nonzero compactly supported cohomology outside degree 0. In degree 0 (i.e. for $l = -1$), it vanishes unless there is a compact connected component, in which case it is one-dimensional. A more detailed study of this region can be found in Section B.2.1, and its upshot is that

$$\mathrm{H}^l\big({}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z}\big) \simeq 0 \text{ for } l \neq -1 \tag{3.7}$$

and

$$\mathrm{H}^{-1}\big({}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z}\big)_{(\check{w},\check{t})} \simeq \begin{cases} \mathbf{k} & \text{if } c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \text{ and } -\varphi^+_{\mathrm{r},c}(\check{w}) \leq \check{t} < -\varphi^-_{\mathrm{r},c}(\check{w}), \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

with the continuous functions $\varphi^+_{\mathrm{r},c}, \varphi^-_{\mathrm{r},c} \colon \mathbb{C}_w \to \mathbb{R}$ defined by

$$\varphi^+_{\mathrm{r},c}(w_1 + iw_2) := \begin{cases} \frac{w_1^2}{2c_1} & \text{if } w_1 \leq 0, \\ 0 & \text{if } w_1 > 0 \end{cases}$$

and

$$\varphi^-_{\mathrm{r},c}(w_1 + iw_2) := \begin{cases} \frac{1}{2|c|^2}(c_1w_1^2 - c_1w_2^2 + 2c_2w_1w_2) = \operatorname{Re}\frac{1}{2c}w^2 & \text{if } w_1 \leq 0, \\ -\frac{1}{2c_1|c|^2}(c_1w_2 - c_2w_1)^2 = \eta_c(w) & \text{if } w_1 > 0. \end{cases}$$

Observing that $\varphi^+_{\mathrm{r},c}(w) - \varphi^-_{\mathrm{r},c}(w) = \frac{1}{2c_1|c|^2}(c_1w_2 - c_2w_1)^2$, we see that $\varphi^+_{\mathrm{r},c}(w) \geq \varphi^-_{\mathrm{r},c}(w)$ for all $w \in \mathbb{C}_w$. We write for short $\eta_c(w) := -\frac{1}{2c_1|c|^2}(c_1w_2 - c_2w_1)^2$ (as we already did above).

We set $\widehat{\mathcal{H}}_- := \{w \in \mathbb{C}_w \mid c_1w_2 - c_2w_1 \leq 0\}$, which is a half-plane whose upper boundary is the line $w_2 = \frac{c_2}{c_1}w_1$. Note that this half-plane only depends on $\arg C$. The stalks suggest the following global statement.

Proposition 3.10. *There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$${}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z} \simeq \mathsf{E}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}[1].$$

Proof. Set $A := \{(z,w,t) \in \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \mid z \in \mathcal{S}_1, t - \operatorname{Re}(zw + \frac{c}{2}z^2) \geq 0\}$, and recall from (3.5) that ${}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z} \simeq \mathrm{R}\widetilde{q}_! \mathbf{k}_A[1]$.

First, consider the set

$$U := \Big\{(z,w,t) \in \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \,\Big|\, z \in \mathcal{S}_1, t - \operatorname{Re}\Big(zw + \frac{c}{2}z^2\Big) \geq 0, t < -\varphi^-_{\mathrm{r},c}(w)\Big\}.$$

It is an open subset of A, and hence there is a short exact sequence in $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}})$

$$0 \longrightarrow \mathbf{k}_U \longrightarrow \mathbf{k}_A \longrightarrow \mathbf{k}_{A\smallsetminus U} \longrightarrow 0,$$

inducing a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$\mathrm{R}\widetilde{q}_!\mathbf{k}_U \longrightarrow \mathrm{R}\widetilde{q}_!\mathbf{k}_A \longrightarrow \mathrm{R}\widetilde{q}_!\mathbf{k}_{A\smallsetminus U} \xrightarrow{+1}.$$

By the projection formula,

$$\mathrm{R}\widetilde{q}_!\mathbf{k}_{A\smallsetminus U} \simeq \mathrm{R}\widetilde{q}_!\big(\mathbf{k}_A \otimes \widetilde{q}^{\,-1}\mathbf{k}_{\{(w,t)\in\mathbb{C}_w\times\mathbb{R}\,|\,t\geq-\varphi^-_{\mathrm{r},c}(w)\}}\big) \simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_A \otimes \mathbf{k}_{\{(w,t)\in\mathbb{C}_w\times\mathbb{R}\,|\,t\geq-\varphi^-_{\mathrm{r},c}(w)\}},$$

and hence it follows from (3.7) and (3.8) that the stalks of the cohomology sheaves of $\mathrm{R}\widetilde{q}_!\mathbf{k}_{A\smallsetminus U}$ at an arbitrary point $(\check{w},\check{t})$ are all zero. Therefore, $\mathrm{R}\widetilde{q}_!\mathbf{k}_{A\smallsetminus U} \simeq 0$, and $\mathrm{R}\widetilde{q}_!\mathbf{k}_A \simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_U$ by the distinguished triangle above.

Next, consider the set

$$B := \Big\{(z,w,t) \in \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R} \,\Big|\, z_1 = \frac{1}{c_1}\Big(\sqrt{2c_1t + w_1^2} - w_1\Big), z_2 = 0, \\ c_1w_2 - c_2w_1 \leq 0, -\varphi^+_{\mathrm{r},c}(w) \leq t < -\varphi^-_{\mathrm{r},c}(w)\Big\}.$$

For fixed $\check{w}$ and $\check{t}$, the corresponding point $z = \frac{1}{c_1}\big(\sqrt{2c_1\check{t} + \check{w}_1^2} - \check{w}_1\big)$ is the rightmost intersection point of the hyperbolic region $\Xi_{\check{w},\check{t}} = \{\check{t} - \operatorname{Re}(z\check{w} + \frac{c}{2}z^2) \geq 0\}$ with the horizontal border of the sector $\mathcal{S}_1$. In particular, it is a point in the compact connected component of $\Xi_{\check{w},\check{t}} \cap \mathcal{S}_1$. (As we have seen before, the conditions on w and t ensure that there is a compact connected component.) Moreover, B is a closed subset of U and we get a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$\mathrm{R}\widetilde{q}_!\mathbf{k}_{U\smallsetminus B} \longrightarrow \mathrm{R}\widetilde{q}_!\mathbf{k}_U \longrightarrow \mathrm{R}\widetilde{q}_!\mathbf{k}_B \xrightarrow{+1}.$$

The stalks of the cohomology sheaves of $\mathrm{R}\widetilde{q}_!\mathbf{k}_{U\smallsetminus B}$ are compactly supported cohomology groups of the spaces $\Xi_{\check{w},\check{t}} \cap \mathcal{S}_1$ minus the rightmost point in the compact connected component (if such a component exists) and as such they all vanish (see Lemma B.1). Consequently, $\mathrm{R}\widetilde{q}_!\mathbf{k}_{U\smallsetminus B} \simeq 0$ and $\mathrm{R}\widetilde{q}_!\mathbf{k}_B \simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_U$.

Finally, consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, hook, "j_B"] & \{(w,t)\in\mathbb{C}_w\times\mathbb{R} \mid c_1w_2-c_2w_1\leq 0, -\varphi^+_{\mathrm{r},c}(w)\leq t< -\varphi^-_{\mathrm{r},c}(w)\} \arrow[d, hook, "j"] \\ \mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R} \arrow[r, "\widetilde{q}"] & \mathbb{C}_w\times\mathbb{R}, \end{tikzcd}$$

where h is the projection onto the last two components (i.e. induced by $\widetilde{q}$). Starting from what we obtained above, we can compute

$$\begin{aligned} {}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z}[-1] &\simeq \mathrm{R}\widetilde{q}_!\mathbf{k}_A \simeq \mathrm{R}\widetilde{q}_! j_{B!}\mathbf{k}_B \simeq j_! h_! \mathbf{k}_B \\ &\overset{(*)}{\simeq} j_!\mathbf{k}_{\{(w,t)\in\mathbb{C}_w\times\mathbb{R} \mid c_1w_2-c_2w_1\geq 0, -\varphi^+_{\mathrm{r},c}(w)\leq t<-\varphi^-_{\mathrm{r},c}(w)\}} \\ &\simeq \pi^{-1}\mathbf{k}_{\widehat{\mathcal{H}}_-}\otimes\mathbf{k}_{\{-\varphi^+_{\mathrm{r},c}\leq t<\varphi^-_{\mathrm{r},c}\}} \simeq \mathsf{E}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}. \end{aligned}$$

The isomorphism $(*)$ follows because h is a homeomorphism (see Lemma A.1), and this concludes the proof. $\square$

The cases of the sectors $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ can be treated analogously. For the sectors $\mathcal{S}_2$ and $\mathcal{S}_3$, one needs to introduce the continuous functions $\varphi^+_{\mathrm{l},c}, \varphi^-_{\mathrm{l},c}\colon \mathbb{C}_w\to\mathbb{R}$, which are given by

$$\varphi^+_{\mathrm{l},c}(w_1+iw_2) := \begin{cases} 0 & \text{if } w_1<0, \\ \frac{w_1^2}{2c_1} & \text{if } w_1\geq 0 \end{cases}$$

and

$$\varphi^-_{\mathrm{l},c}(w_1+iw_2) := \begin{cases} \eta_c(w) & \text{if } w_1<0, \\ \operatorname{Re}\frac{1}{2c}w^2 & \text{if } w_1\geq 0. \end{cases}$$

Compared to the definition of $\varphi^+_{\mathrm{r},c}$ and $\varphi^-_{\mathrm{r},c}$, the cases are interchanged. Furthermore, we set $\widehat{\mathcal{H}}_+ := \{w\in\mathbb{C}_w \mid c_1w_2-c_2w_1\geq 0\}$ and obtain the following result.

Proposition 3.11. *There are isomorphisms in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$\begin{aligned} {}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_2|\mathbb{C}_z} &\simeq \mathsf{E}^{\varphi^+_{\mathrm{l},c}\rhd\varphi^-_{\mathrm{l},c}}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}[1], \\ {}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_3|\mathbb{C}_z} &\simeq \mathsf{E}^{\varphi^+_{\mathrm{l},c}\rhd\varphi^-_{\mathrm{l},c}}_{\widehat{\mathcal{H}}_+|\mathbb{C}_w}[1], \\ {}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_4|\mathbb{C}_z} &\simeq \mathsf{E}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}}_{\widehat{\mathcal{H}}_+|\mathbb{C}_w}[1]. \end{aligned}$$

Exponentials on the half-lines $\mathcal{S}_{k,k+1}$

The next step is to treat the case of $\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{k,k+1}|\mathbb{C}_z}$. The stalks of its enhanced Fourier–Sato transform at $(\check{w},\check{t})$ are given by the compactly supported cohomology groups of the space $\Xi_{\check{w},\check{t}}\cap\mathcal{S}_{k,k+1}$. The topology of this intersection is studied in Section B.2.2. For example, for $k=1$, we find that the cohomology groups vanish except for degree -1, where they are nonzero if and only if $c_1\check{w}_2 - c_2\check{w}_1 \leq 0$ and

$$0 \leq \check{t} < \frac{1}{2c_1|c|^2}(c_1\check{w}_2 - c_2\check{w}_1)^2.$$

This leads us to the idea that the transforms are again (shifted) exponential enhanced sheaves.

The following proposition summarizes the results for all $k \in \mathbb{Z}/4\mathbb{Z}$. The proof is analogous to that of Proposition 3.10.

Proposition 3.12. *There are isomorphisms in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$\begin{aligned}
{}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{12}|\mathbb{C}_z} &\simeq \mathsf{E}^{0\rhd\eta_c}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}[1],\\
{}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{23}|\mathbb{C}_z} &\simeq \mathsf{E}^{\varphi^+_{\mathrm{l},c}}_{\mathbb{C}_w|\mathbb{C}_w}[1],\\
{}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{34}|\mathbb{C}_z} &\simeq \mathsf{E}^{0\rhd\eta_c}_{\widehat{\mathcal{H}}_+|\mathbb{C}_w}[1],\\
{}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{41}|\mathbb{C}_z} &\simeq \mathsf{E}^{\varphi^+_{\mathrm{r},c}}_{\mathbb{C}_w|\mathbb{C}_w}[1].
\end{aligned}$$

Exponential on the origin

In order to compute the enhanced Fourier–Sato transform of $\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}|\mathbb{C}_z}$, we could proceed similarly. However, in this case there is also a different approach using a result about the classical Fourier–Sato transform for conic sheaves (see [21, Section 3.7]) and its compatibility with the enhanced version (see [26]).

Consider the functor $\varepsilon\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}\times\mathbb{R}}), F \mapsto \pi^{-1}F \otimes \mathbf{k}_{\{t\geq 0\}}$ and note that the enhanced sheaf to which we want to apply the enhanced Fourier–Sato transform can be written as

$$\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}|\mathbb{C}} \simeq \mathsf{E}^{0}_{\{0\}|\mathbb{C}} \simeq \pi^{-1}\mathbf{k}_{\{0\}} \otimes \mathbf{k}_{\{t\geq 0\}} \simeq \varepsilon(\mathbf{k}_{\{0\}}).$$

We denote by $(\bullet)^\wedge$ the classical Fourier–Sato transform introduced in [21]. Since $\mathbf{k}_{\{0\}} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}})$ is a conic sheaf, we can use [26, Theorem 5.7] and obtain[6]

$${}^{\mathcal{L}}\varepsilon(\mathbf{k}_{\{0\}}) \simeq \varepsilon\big((\mathbf{k}_{\{0\}})^\wedge\big)[1].$$

The main step is therefore the computation of the classical Fourier–Sato transform of $\mathbf{k}_{\{0\}}$. Denoting by $f\colon \{\mathrm{pt}\} \hookrightarrow \mathbb{C}_z, \mathrm{pt} \mapsto 0$ the inclusion of the origin and using [21, Proposition 3.7.14], one gets

$$(\mathbf{k}_{\{0\}})^\wedge \simeq (\mathrm{R}f_!\mathbf{k}_{\{\mathrm{pt}\}})^\wedge \simeq {}^{\mathrm{t}}f^{-1}\big((\mathbf{k}_{\{\mathrm{pt}\}})^\wedge\big),$$

where ${}^{\mathrm{t}}f\colon \mathbb{C}_w \to \{\mathrm{pt}\}$ is the obvious map. (Note that we identify $\mathbb{C}_w$ with the dual of

[6] The statement of [26, Theorem 5.7] is formulated in the context of enhanced ind-sheaves. However, the authors actually prove the statement for enhanced sheaves, which is what we use here. The shift is due to the fact that we defined the enhanced Fourier–Sato transform with a shift, contrarily to the authors of [26].

$\mathbb{C}_z$ by the natural multiplication of complex numbers.) It is not difficult to see from the definition of the classical Fourier–Sato transform that $(\mathbf{k}_{\{\mathrm{pt}\}})^\wedge \simeq \mathbf{k}_{\{\mathrm{pt}\}}$ and hence

$$(\mathbf{k}_{\{0\}})^\wedge \simeq \mathbf{k}_{\mathbb{C}_w}.$$

Altogether, we obtain the following result.

Proposition 3.13. *There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$${}^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}|\mathbb{C}_z} \simeq \mathsf{E}^0_{\mathbb{C}_w|\mathbb{C}_w}[1].$$

Morphisms between exponentials

In the course of our considerations, we will need to apply the enhanced Fourier–Sato transform not only to exponential enhanced sheaves but also to morphisms between them. Recall that (writing out convolution), we have

$${}^{\mathcal{L}}\mathcal{F} \simeq \mathrm{R}\widetilde{q}_!\mathrm{R}\mu_!\big(q_1^{-1}\mathsf{E}^{-zw}_{\mathbb{C}_z\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \otimes q_2^{-1}\widetilde{p}^{-1}\mathcal{F}\big)[1]$$

with the maps $q_1, q_2, \mu\colon \mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}^2 \to \mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}$ defined by $q_1(z,w,t_1,t_2) = (z,w,t_1)$, $q_2(z,w,t_1,t_2) = (z,w,t_2)$ and $\mu(z,w,t_1,t_2) = (z,w,t_1+t_2)$.

For example, assume the morphism $\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z} \to \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{12}|\mathbb{C}_z}$ to be given by an element $\alpha\in\mathbf{k}$ (cf. Lemma 2.8), i.e. on stalks it is multiplication by α.

Applying the functor $\widetilde{p}^{-1}$ yields a morphism $\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_1\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w} \to \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{12}\times\mathbb{C}_w|\mathbb{C}_z\times\mathbb{C}_w}$ still given by α.

Likewise, after pulling back via q_2 and tensoring with the exponent $-zw$, we still have constant sheaves on subspaces of $\mathbb{C}_z\times\mathbb{C}_w\times\mathbb{R}^2$ and the morphism is given by α.

Finally, we apply the proper direct image functors along the maps μ (sum of the real variables, i.e. a projection not parallel to a coordinate axis) and $\widetilde{q}$. They are subfunctors of the (usual, i.e. non-proper) direct image functors, and since sections of direct images are given by sections of the original sheaf, we end up with the morphism $\mathsf{E}^{\varphi^{-}_{\mathrm{r},d}\rhd\varphi^{-}_{\mathrm{r},d}}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w} \to \mathsf{E}^{0\rhd\eta_c}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}$ given by α. (In the end, one needs to shift everything by 1.)

Accordingly, we can see that a morphism between a direct sum of exponentials as above (given by a matrix) is transformed into a morphism between the direct sum of the enhanced Fourier–Sato transforms (which are again shifted exponentials) given by the same matrix.

Remark. It is worth pointing out that a morphism $\mathsf{E}^{\varphi^{-}_{\mathrm{r},c}\rhd\varphi^{-}_{\mathrm{r},c}}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w} \to \mathsf{E}^{0\rhd\eta_c}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}$ is indeed given by an element $\alpha\in\mathbf{k}$. This is shown in Lemma A.4.

3.3.4. The enhanced Fourier–Sato transform of a Gaussian enhanced sheaf

In this section, we will elaborate on the idea given at the end of Section 3.3.1 in order to describe the enhanced Fourier–Sato transform of $\mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma}$. We write for short $\mathcal{F} := \mathcal{F}^{C,\theta_0,\mathbb{r}}_{\sigma}$.

To make notation easier, we will write E_Z^{φ} instead of $\mathsf{E}_{Z|X}^{\varphi}$ since the second index will always be $X = \mathbb{C}_z$ or $X = \mathbb{C}_w$ and it is clear from the context whether the local variable is z or w. (The same applies to the other type of exponential enhanced sheaf.) Furthermore, we shall not write the exponents r_c, so we will essentially assume $r_c = 1$ for any $c \in C$. This is, however, not a restriction but only a notational modification for the sake of better readability: One can replace any occurence of a direct sum $\bigoplus\limits_{c\in C} \mathsf{E}_Z^{\varphi_c}$ by $\bigoplus\limits_{c\in C} \big(\mathsf{E}_Z^{\varphi_c}\big)^{r_c}$ (and similarly for the other type of exponential) and the word "triangular matrix" by "block-triangular matrix". The proof will then apply in the exact same form.

Recall that we chose the generic direction $\theta_0 = -\frac{1}{2}\arg C$. It is easy to check that the order defined by $c_{(1)} <_{\theta_0} \ldots <_{\theta_0} c_{(n)}$ on the elements of C is the order given by $|c_{(1)}| < \ldots < |c_{(n)}|$: For $z = |z|e^{-\frac{1}{2}i\arg C}$, $c = |c|e^{i\arg C}$ and $d = |d|e^{i\arg C}$, the inequality $\operatorname{Re}\frac{c}{2}z^2 < \operatorname{Re}\frac{d}{2}z^2$ is equivalent to $|c| < |d|$.

Recall moreover that we have defined a covering of the plane $\mathbb{C}_z$ by four closed sectors $\mathcal{S}_k$, $k \in \mathbb{Z}/4\mathbb{Z}$. We set $\mathcal{H}_+ := \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{H}_- := \mathcal{S}_3 \cup \mathcal{S}_4$ as well as $\mathcal{S}_{k,k+1} := \mathcal{S}_k \cap \mathcal{S}_{k+1}$. On these sectors, we have isomorphisms

$$\alpha_k \colon \mathcal{F}_{\mathcal{S}_k} \xrightarrow{\sim} \bigoplus_{c\in C} \mathsf{E}_{\mathcal{S}_k}^{-\operatorname{Re}\frac{c}{2}z^2},$$

and the gluing morphisms $\alpha_{k+1}\circ\alpha_k^{-1}$ on $\mathcal{S}_{k,k+1}$ are given by the Stokes multipliers σ_k. The matrices σ_1 and σ_3 are upper triangular, and the matrices σ_2 and σ_4 are lower triangular. (Since the structure of these matrices depends on the order on C, one needs to read any direct sum indexed over C as a direct sum indexed over $\{1, \ldots, n\}$ throughout the proof.)

In the sequel, we will use the isomorphisms

$$\mathcal{F}_{\mathcal{S}_{k,k+1}} \xrightarrow{\sim} \bigoplus_{c\in C} \mathsf{E}_{\mathcal{S}_{k,k+1}}^{-\operatorname{Re}\frac{c}{2}z^2}$$

induced by α_{k+1} (i.e. on a half-line between two sectors, we will use the isomorphism induced by the one on the sector following in clockwise direction). We will also use the isomorphism

$$\mathcal{F}_{\{0\}} \xrightarrow{\sim} \bigoplus_{c\in C} \mathsf{E}_{\{0\}}^{-\operatorname{Re}\frac{c}{2}z^2} = \bigoplus_{c\in C} \mathsf{E}_{\{0\}}^{0}$$

induced by α_1.

Transform of restrictions to half-planes

Let us start by investigating the short exact sequence in $\operatorname{Mod}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}})$

$$0 \longrightarrow \mathcal{F}_{\mathcal{H}_+} \longrightarrow \mathcal{F}_{\mathcal{S}_1} \oplus \mathcal{F}_{\mathcal{S}_2} \longrightarrow \mathcal{F}_{\mathcal{S}_{12}} \longrightarrow 0. \tag{3.9}$$

We know that $\mathcal{F}_{\mathcal{S}_1}$ and $\mathcal{F}_{\mathcal{S}_2}$ are isomorphic to the direct sum of exponentials via α_1 and α_2, and we also have such an isomorphism for $\mathcal{F}_{\mathcal{S}_{12}}$ (as stated above). Therefore, the short exact sequence (3.9) is isomorphic to

$$0 \longrightarrow \mathcal{F}_{\mathcal{H}_+} \longrightarrow \bigoplus_{c\in C} \mathsf{E}_{\mathcal{S}_1}^{-\operatorname{Re}\frac{c}{2}z^2} \oplus \bigoplus_{c\in C} \mathsf{E}_{\mathcal{S}_2}^{-\operatorname{Re}\frac{c}{2}z^2} \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{c\in C} \mathsf{E}_{\mathcal{S}_{12}}^{-\operatorname{Re}\frac{c}{2}z^2} \longrightarrow 0.$$

Applying the enhanced Fourier–Sato transform and a shift by -1 and using the results of the previous section, we get a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})[-1] \longrightarrow \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c}\rhd\varphi^-_{\mathrm{l},c}} \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0\rhd\eta_c} \xrightarrow{+1} . \tag{3.10}$$

It induces a long exact sequence of the cohomologies in $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$. Since the second and third object are concentrated in degree 0, it follows that $\mathrm{H}^l({}^{\mathcal{L}}\mathcal{F}_{\mathcal{H}_+}) \simeq 0$ for $l \notin \{-1, 0\}$. Therefore, the long exact sequence can contain at most four nonzero objects, and it reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{-1}\big({}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})\big) \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c}\rhd\varphi^-_{\mathrm{l},c}} \arrow[dl, "\sigma_1 - \mathbb{1}"] & \\
& \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0\rhd\eta_c} \arrow[r] & \mathrm{H}^{0}\big({}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})\big) \arrow[r] & 0.
\end{tikzcd} \tag{3.11}$$

Lemma 3.14. *The morphism $\sigma_1 - \mathbb{1}$ from (3.11) is an epimorphism in* $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$.

Proof. It suffices to check surjectivity on stalks. Let therefore $(\check{w}, \check{t}) \in \mathbb{C}_w \times \mathbb{R}$ be a point with $\check{w}_1 \geq 0$, $\check{w} \in \widehat{\mathcal{H}}_-$ and $\check{t} \geq 0$.

The stalk at $(\check{w}, \check{t})$ of the morphism

$$\bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c}\rhd\varphi^-_{\mathrm{l},c}} \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0\rhd\eta_c}$$

is the linear map

$$\begin{gathered}
\bigoplus_{j=1}^{n} \bigstar_j \oplus \bigoplus_{j=1}^{n} \blacktriangle_j \longrightarrow \bigoplus_{j=1}^{n} \bigstar_j, \\
\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \longmapsto \sigma_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.
\end{gathered}$$

Here $\bigstar_j = \mathbf{k}$ if $\check{t} < -\eta_{c_{(j)}}(\check{w})$, and $\bigstar_j = 0$ otherwise. Similarly, $\blacktriangle_j = \mathbf{k}$ if $\check{t} < -\operatorname{Re}\frac{1}{2c_{(j)}}\check{w}^2$,

and $\blacktriangle_j = 0$ otherwise. It is surjective since a preimage of $(\gamma_1, \dots, \gamma_n)^{\mathrm{T}}$ is given by $\big(\sigma_1^{-1}(\gamma_1, \dots, \gamma_n)^{\mathrm{T}}, (0, \dots, 0)^{\mathrm{T}}\big)$.
(Let us make clear that this map is well-defined: Note that $d = \lambda c$ (with $\lambda \in \mathbb{R}_{>0}$) implies $\lambda\eta_d = \eta_c$, so we have $0 < -\eta_{c_{(n)}} < \dots < -\eta_{c_{(1)}}$. Therefore, $\bigstar_l = 0$ implies $\bigstar_j = 0$ for all $j \geq l$. Moreover, recall that we have $-\eta_{c_{(j)}}(w) \geq -\operatorname{Re} \frac{1}{2c_{(j)}} w^2$ for any w. Therefore, $\blacktriangle_j = 0$ whenever $\bigstar_j = 0$. Together with the fact that σ_1 is upper triangular, this shows that the map is well-defined.)

The case where $\breve{w}_1 < 0$ but still $\breve{w} \in \widehat{\mathcal{H}}_-$ and $\check{t} \geq 0$ works similarly. If $\breve{w} \notin \widehat{\mathcal{H}}_-$ or $\check{t} < 0$, the stalk of the target sheaf is zero, so the induced map on stalks is also surjective. $\square$

The following proposition is an immediate consequence of the previous lemma.

Proposition 3.15. *The complex* ${}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})$ *is concentrated in degree* -1*. More precisely, there is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$

$$ {}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})[-1] \simeq \ker\left(\sigma_1 - \mathbb{1} \colon \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c} \rhd \varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c} \rhd \varphi^-_{\mathrm{l},c}} \longrightarrow \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta_c}\right). $$

Proof. Let us denote the kernel in the statement of the proposition by $K \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$. Clearly, we have a short exact sequence in $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$

$$ 0 \longrightarrow K \longrightarrow \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c} \rhd \varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c} \rhd \varphi^-_{\mathrm{l},c}} \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta_c} \longrightarrow 0, $$

inducing a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$ (where one regards the objects as complexes sitting in degree 0).

Since the second and third objects are the same as in (3.10), one has identity morphisms between these objects, and by the axioms of a triangulated category this extends to an isomorphism of distinguished triangles, yielding the dashed isomorphism as desired:

$$ \begin{tikzcd} {}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})[-1] \arrow[r] \arrow[d, dashed, "\simeq"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c} \rhd \varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c} \rhd \varphi^-_{\mathrm{l},c}} \arrow[r, "\sigma_1 - \mathbb{1}"] \arrow[d, equal] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta_c} \arrow[r, "+1"] \arrow[d, equal] & {} \\ K \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{r},c} \rhd \varphi^-_{\mathrm{r},c}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi^+_{\mathrm{l},c} \rhd \varphi^-_{\mathrm{l},c}} \arrow[r, "\sigma_1 - \mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta_c} \arrow[r, "+1"] & {} \end{tikzcd} . $$

In general, the morphism completing such a morphism of distinguished triangles is not unique. However, in this case we can prove uniqueness via [11, Corollary IV.1.5], i.e. we have to check that

$$ \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})}\Big(K, \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta_c}[-1]\Big) \simeq 0. $$

Since all the objects are concentrated in degree 0, this Hom-set is nothing but

$$\mathrm{Ext}^{-1}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})}\Big(K, \bigoplus_{c\in C} \mathsf{E}^{0\rhd\eta_c}_{\widehat{\mathcal{H}}_-}\Big),$$

which vanishes like any Ext group of negative degree (cf. [11, Theorem III.5.5]). □

Proceeding analogously with the sequence

$$0 \longrightarrow \mathcal{F}_{\mathcal{H}_-} \longrightarrow \mathcal{F}_{\mathcal{S}_4} \oplus \mathcal{F}_{\mathcal{S}_3} \longrightarrow \mathcal{F}_{\mathcal{S}_{34}} \longrightarrow 0, \tag{3.12}$$

we get a similar result.

Proposition 3.16. *The complex* ${}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_-})$ *is concentrated in degree* -1*. More precisely, there is an isomorphism in* $\mathrm{D^b}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$${}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_-})[-1] \simeq \ker\left(\mathbb{1} - \sigma_3\colon \bigoplus_{c\in C} \mathsf{E}^{\varphi^+_{\mathrm{r},c}\rhd\varphi^-_{\mathrm{r},c}}_{\widehat{\mathcal{H}}_+} \oplus \bigoplus_{c\in C} \mathsf{E}^{\varphi^+_{\mathrm{l},c}\rhd\varphi^-_{\mathrm{l},c}}_{\widehat{\mathcal{H}}_+} \longrightarrow \bigoplus_{c\in C} \mathsf{E}^{0\rhd\eta_c}_{\widehat{\mathcal{H}}_+}\right).$$

Transform of restriction to real axis

We treat the sequence

$$0 \longrightarrow \mathcal{F}_L \longrightarrow \mathcal{F}_{\mathcal{S}_{41}} \oplus \mathcal{F}_{\mathcal{S}_{23}} \longrightarrow \mathcal{F}_{\{0\}} \longrightarrow 0 \tag{3.13}$$

similarly to the sequences in the previous subsection: First, we can make it more explicit by using the trivializing isomorphisms and obtain

$$0 \longrightarrow \mathcal{F}_L \longrightarrow \bigoplus_{c\in C} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{41}} \oplus \bigoplus_{c\in C} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{23}} \xrightarrow{\mathbb{1}-\sigma_4\sigma_3} \bigoplus_{c\in C} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}} \longrightarrow 0.$$

The enhanced Fourier–Sato transform then yields a distinguished triangle which reduces to a short exact sequence, finally giving the following result.

Proposition 3.17. *The complex* ${}^{\mathcal{L}}(\mathcal{F}_L)$ *is concentrated in degree* -1*. More precisely, there is an isomorphism in* $\mathrm{D^b}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$${}^{\mathcal{L}}(\mathcal{F}_L)[-1] \simeq \ker\left(\mathbb{1} - \sigma_4\sigma_3\colon \bigoplus_{c\in C} \mathsf{E}^{\varphi^+_{\mathrm{r},c}}_{\mathbb{C}_w} \oplus \bigoplus_{c\in C} \mathsf{E}^{\varphi^+_{\mathrm{l},c}}_{\mathbb{C}_w} \longrightarrow \bigoplus_{c\in C} \mathsf{E}^{0}_{\mathbb{C}_w}\right).$$

Transform on the whole plane

We can now start to examine the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{\mathcal{H}_+} \oplus \mathcal{F}_{\mathcal{H}_-} \longrightarrow \mathcal{F}_L \longrightarrow 0, \tag{3.14}$$

which will enable us to describe ${}^{\mathcal{L}}\mathcal{F}$ and show that it is of the desired form on sectors.

Let us first understand the morphism $\mathcal{F}_{\mathcal{H}_+} \oplus \mathcal{F}_{\mathcal{H}_-} \to \mathcal{F}_L$. It is the difference of the natural morphisms $\mathcal{F}_{\mathcal{H}_+} \to \mathcal{F}_L$ and $\mathcal{F}_{\mathcal{H}_-} \to \mathcal{F}_L$, which can be described more explicitly. To do this, note that the sequences (3.9) and (3.13) fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\mathcal{H}_+} \arrow[r] \arrow[d] & \mathcal{F}_{\mathcal{S}_1} \oplus \mathcal{F}_{\mathcal{S}_2} \arrow[r] \arrow[d] & \mathcal{F}_{\mathcal{S}_{12}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_L \arrow[r] & \mathcal{F}_{\mathcal{S}_{41}} \oplus \mathcal{F}_{\mathcal{S}_{23}} \arrow[r] & \mathcal{F}_{\{0\}} \arrow[r] & 0,
\end{tikzcd}$$

which is isomorphic to

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\mathcal{H}_+} \arrow[r] \arrow[d] & \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_1}^{-\operatorname{Re}\frac{c}{2}z^2} \oplus \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_2}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r, "\sigma_1 - \mathbb{1}"] \arrow[d, "\mathbb{1} | \sigma_2"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_{12}}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r] \arrow[d, "\sigma_1^{-1}"] & 0 \\
0 \arrow[r] & \mathcal{F}_L \arrow[r] & \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_{41}}^{-\operatorname{Re}\frac{c}{2}z^2} \oplus \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_{23}}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r, "\mathbb{1} - \sigma_4\sigma_3"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\{0\}}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r] & 0.
\end{tikzcd} \tag{3.15}$$

Here, the vertical arrow in the middle denotes the direct sum of the morphism given by the matrix $\mathbb{1}$ between the left summands and the morphism given by the matrix σ_2 between the right summands. We will denote this morphism by $\mathbb{1}|\sigma_2$ (with a bar in case there is no vertical arrow). Since the diagram has exact rows, the first vertical arrow is induced by the second.

A similar diagram exists for the sequences (3.12) and (3.13):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\mathcal{H}_-} \arrow[r] \arrow[d] & \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_4}^{-\operatorname{Re}\frac{c}{2}z^2} \oplus \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_3}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r, "\mathbb{1} - \sigma_3"] \arrow[d, "\sigma_4 | \mathbb{1}"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_{34}}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r] \arrow[d, "\sigma_4"] & 0 \\
0 \arrow[r] & \mathcal{F}_L \arrow[r] & \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_{41}}^{-\operatorname{Re}\frac{c}{2}z^2} \oplus \bigoplus\limits_{c\in C} \mathsf{E}_{\mathcal{S}_{23}}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r, "\mathbb{1} - \sigma_4\sigma_3"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\{0\}}^{-\operatorname{Re}\frac{c}{2}z^2} \arrow[r] & 0.
\end{tikzcd} \tag{3.16}$$

Let us now describe ${}^{\mathcal{L}}\mathcal{F}$. The considerations from the previous sections suggest that a decomposition of $\mathbb{C}_w$ into four sectors with boundaries given by the lines $w_1 = 0$ and $c_1 w_2 - c_2 w_1 = 0$ might suit our computations. Therefore, we define the sectors (for $\arg C \in (-\frac{\pi}{2}, \frac{\pi}{2})$)

$$\begin{aligned}
\widehat{\mathcal{S}}_1 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[-\pi + \arg C, -\frac{\pi}{2}\right] \text{ if } w \neq 0 \right\}, \\
\widehat{\mathcal{S}}_2 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[-\frac{\pi}{2}, \arg C\right] \text{ if } w \neq 0 \right\}, \\
\widehat{\mathcal{S}}_3 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[\arg C, \frac{\pi}{2}\right] \text{ if } w \neq 0 \right\},
\end{aligned}$$

$$\widehat{\mathcal{S}}_4 := \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[\frac{\pi}{2}, \pi + \arg C\right] \text{ if } w \neq 0 \right\}.$$

(An a posteriori justification for the numbering of these sectors is given by Proposition 3.20.) We have $\widehat{\mathcal{H}}_+ = \widehat{\mathcal{S}}_3 \cup \widehat{\mathcal{S}}_4$ and $\widehat{\mathcal{H}}_- = \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$. As usual, we set $\widehat{\mathcal{S}}_{k,k+1} := \widehat{\mathcal{S}}_k \cap \widehat{\mathcal{S}}_{k+1}$.

Proposition 3.18. *On the sectors $\widehat{\mathcal{S}}_k$ (more precisely, on $\widehat{\mathcal{S}}_k \times \mathbb{R}$), the enhanced Fourier–Sato transform of $\mathcal{F}$ is isomorphic to a direct sum of exponential enhanced sheaves with exponential factors $\operatorname{Re} \frac{1}{2c} w^2$. In particular, ${}^{\mathcal{L}}\mathcal{F}$ is concentrated in degree 0, and for every $k \in \mathbb{Z}/4\mathbb{Z}$ we have an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$*

$$({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_k} \simeq \bigoplus_{c \in C} \mathsf{E}^{\operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_k}.$$

Proof. We will prove the desired isomorphism for $k = 1$. The other cases are similar.

From the sequence (3.14), we get a distinguished triangle

$${}^{\mathcal{L}}\mathcal{F} \longrightarrow {}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+}) \oplus {}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_-}) \longrightarrow {}^{\mathcal{L}}(\mathcal{F}_L) \xrightarrow{+1}.$$

It is isomorphic to a distinguished triangle

$$\ker(\sigma_1 - \mathbb{1}) \oplus \ker(\mathbb{1} - \sigma_3) \xrightarrow{(\mathbb{1}|\sigma_2) - (\sigma_4|\mathbb{1})} \ker(\mathbb{1} - \sigma_4\sigma_3) \longrightarrow {}^{\mathcal{L}}\mathcal{F} \xrightarrow{+1}. \tag{3.17}$$

Here, the kernels are the ones from Propositions 3.15, 3.16 and 3.17. The first morphism is induced by the ones described in (3.15) and (3.16).

Recall that the functor $(\bullet)_{\widehat{\mathcal{S}}_1}$ is exact, so it commutes with kernels and cohomology. We apply it to the triangle (3.17) to obtain

$$\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \oplus \ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1} \xrightarrow{(\mathbb{1}|\sigma_2) - (\sigma_4|\mathbb{1})} \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \longrightarrow ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1} \xrightarrow{+1}, \tag{3.18}$$

and from the associated long exact sequence it follows that $\mathrm{H}^l\big(({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1}\big) \simeq 0$ for $l \notin \{-1, 0\}$ and that the sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^{-1}\big(({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1}\big) \longrightarrow \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \oplus \ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1} \\ \xrightarrow{(\mathbb{1}|\sigma_2) - (\sigma_4|\mathbb{1})} \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \longrightarrow \mathrm{H}^0\big(({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1}\big) \longrightarrow 0. \end{aligned} \tag{3.19}$$

is exact. It turns out that these kernels can be described more explicitly on a sector.

Firstly, we note that $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1}$ is the kernel of the morphism

$$\bigoplus_{c \in C} \mathsf{E}^{\varphi^+_{\mathrm{r},c} \rhd \varphi^-_{\mathrm{r},c}}_{\widehat{\mathcal{S}}_{41}} \oplus \bigoplus_{c \in C} \mathsf{E}^{\varphi^+_{\mathrm{l},c} \rhd \varphi^-_{\mathrm{l},c}}_{\widehat{\mathcal{S}}_{41}} \xrightarrow{\mathbb{1} - \sigma_3} \bigoplus_{c \in C} \mathsf{E}^{0 \rhd \eta_c}_{\widehat{\mathcal{S}}_{41}},$$

which is obtained by tensoring the morphism from Proposition 3.16 with $\pi^{-1}\mathbf{k}_{\widehat{\mathcal{S}}_1}$ (since

$\widehat{\mathcal{H}}_+ \cap \widehat{\mathcal{S}}_1 = \widehat{\mathcal{S}}_{41}$). Moreover, on $\widehat{\mathcal{S}}_{41}$, we have $c_1 w_2 - c_2 w_1 = 0$, which implies $\varphi^+_{\mathrm{r},c}(w) = \varphi^-_{\mathrm{r},c}(w)$ and $\varphi^+_{\mathrm{l},c}(w) = \varphi^-_{\mathrm{l},c}(w)$. Therefore, every single exponential enhanced sheaf appearing in the direct sums above is zero, and it follows that $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1} \simeq 0$.

Secondly, we determine $\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1}$ similarly: It is the kernel of

$$\bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \oplus \bigoplus_{c\in C} \mathsf{E}^{0\triangleright \eta_c}_{\widehat{\mathcal{S}}_1} \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{c\in C} \mathsf{E}^{0\triangleright \eta_c}_{\widehat{\mathcal{S}}_1}$$

since $w_1 \leq 0$ on $\widehat{\mathcal{S}}_1$. We claim that

$$\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \simeq \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1}. \tag{3.20}$$

To show this, it is enough to prove that the sequence

$$0 \longrightarrow \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \xrightarrow{(\mathbb{1},\sigma_1)} \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \oplus \bigoplus_{c\in C} \mathsf{E}^{0\triangleright \eta_c}_{\widehat{\mathcal{S}}_1} \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{c\in C} \mathsf{E}^{0\triangleright \eta_c}_{\widehat{\mathcal{S}}_1} \longrightarrow 0 \tag{3.21}$$

is exact. Note that the map $(\mathbb{1}, \sigma_1)$ is well-defined since $-\operatorname{Re}\frac{1}{2d}w^2 < -\eta_c(w)$ for $|d| > |c|$ (cf. Lemma A.4). On the contrary, a morphism given by $(\sigma_1^{-1}, \mathbb{1})$ would not be well-defined (and it is the other way round for $\mathcal{S}_2$). Exactness can be checked on stalks: Let $(\check{w}, \check{t}) \in \widehat{\mathcal{S}}_1 \times \mathbb{R}$ be a point. If $\check{t} < 0$, the induced sequence on stalks is

$$0 \longrightarrow V \xrightarrow{(\mathrm{id},0)} V \oplus 0 \longrightarrow 0 \longrightarrow 0,$$

where V is a $\mathbf{k}$-vector space depending on the exact value of $\check{w}$ and $\check{t}$. This sequence is obviously exact.

If $\check{t} \geq 0$, the induced sequence on stalks is

$$0 \longrightarrow \bigoplus_{j=1}^{n} \bigstar_j \xrightarrow{(\mathbb{1},\sigma_1)} \bigoplus_{j=1}^{n} \bigstar_j \oplus \bigoplus_{j=1}^{n} \blacktriangle_j \xrightarrow{\sigma_1 - \mathbb{1}} \bigoplus_{j=1}^{n} \blacktriangle_j \longrightarrow 0,$$

where $\bigstar_j = \mathbf{k}$ if $\check{t} < -\operatorname{Re}\frac{1}{2c_{(j)}}\check{w}^2$, and $\bigstar_j = 0$ otherwise. Accordingly, $\blacktriangle_j = \mathbf{k}$ if $\check{t} < -\eta_{c_{(j)}}(\check{w})$, and $\blacktriangle_j = 0$ otherwise. Note that the morphism

$$\begin{aligned} \sigma_1 \colon \bigoplus_{j=1}^{n} \bigstar_j &\longrightarrow \bigoplus_{j=1}^{n} \blacktriangle_j, \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} &\longmapsto \sigma_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{aligned}$$

appearing twice in this sequence is well-defined since it follows from $\blacktriangle_l = 0$ that $\blacktriangle_j = 0$ for all $j \geq l$, and hence $\bigstar_j = 0$ for all $j \geq l$ (recall that σ_1 is upper triangular). The sequence is exact because an element of the vector space in the middle is in the kernel of $\sigma_1 - \mathbb{1}$ if and only if it is of the form $(v, \sigma_1 v)$ for $v \in \bigoplus_{j=1}^n \bigstar_j$. Injectivity of $(\mathbb{1}, \sigma_1)$ and surjectivity of $\sigma_1 - \mathbb{1}$ are obvious. We have thus proved (3.20).

Thirdly, we proceed analogously to find out that

$$\ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}}, \tag{3.22}$$

which follows from the short exact sequence

$$0 \longrightarrow \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}} \xrightarrow{(\mathbb{1},\sigma_2\sigma_1)} \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{0} \xrightarrow{\mathbb{1}-\sigma_4\sigma_3} \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{0} \longrightarrow 0. \tag{3.23}$$

(Recall that $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$.)

Finally, we realize that there is a commutative diagram in which the sequences (3.21) and (3.23) appear as the columns, and which has exact rows and columns (see p. 75). Comparing the upper row of this diagram with the sequence (3.19), which we wanted to study, it follows that

$$\mathrm{H}^{-1}\big(({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1}\big) \simeq 0 \quad \text{and} \quad \mathrm{H}^0\big(({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1}\big) \simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\mathrm{Re}\,\frac{1}{2c}w^2}.$$

In particular, having proven this analogously for the other $\widehat{\mathcal{S}}_k$, it follows that ${}^{\mathcal{L}}\mathcal{F}$ is concentrated in degree 0 due to the exactness of taking stalks.

Concretely, the desired isomorphism is now obtained by viewing the upper row of the diagram as a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$ and constructing an isomorphism to the triangle (3.18) using the isomorphisms (3.20) and (3.22):

$$\begin{tikzcd}
\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[d, "\simeq"] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1} \arrow[r, "+1"] \arrow[d, dashed, "\simeq"] & {} \\
\bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}\triangleright\mathrm{Re}\,\frac{1}{2c}w^2} \arrow[r, "\mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}} \arrow[r, "\mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\mathrm{Re}\,\frac{1}{2c}w^2} \arrow[r, "+1"] & {}
\end{tikzcd}$$

The dashed isomorphism is then unique in this case since the assumptions of [11, Corollary IV.1.5] are satisfied: One has

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})}\left(\bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}\triangleright\mathrm{Re}\,\frac{1}{2c}w^2}, \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\mathrm{Re}\,\frac{1}{2c}w^2}[-1]\right) \simeq 0$$

because there are no Ext groups of negative degree (cf. [11, Theorem III.5.5]). $\square$

Diagram for $\widehat{\mathcal{S}}_1$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \arrow[r, "\mathbb{1}"] \arrow[d, "(\mathbb{1}{,}\sigma_1)"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}} \arrow[r, "\mathbb{1}"] \arrow[d, "(\mathbb{1}{,}\sigma_2\sigma_1)"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\operatorname{Re} \frac{1}{2c} w^2} \arrow[r] & 0 \\
 & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{0 \triangleright \eta_c} \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[d, "\sigma_1 - \mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{0} \arrow[d, "\mathbb{1} - \sigma_4\sigma_3"] & & \\
 & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{0 \triangleright \eta_c} \arrow[r, "\sigma_1^{-1}"] \arrow[d] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{0} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Remark. As mentioned, the proofs on the other sectors are similar. More precisely, one obtains

$$\begin{aligned}
\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_2} &\simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\, \frac{1}{2c} w^2}, \\
\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_2} &\simeq 0, \\
\ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2} &\simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1}};
\end{aligned}$$

$$\begin{aligned}
\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_3} &\simeq 0, \\
\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_3} &\simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_3}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\, \frac{1}{2c} w^2}, \\
\ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_3} &\simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_3}^{\frac{w_1^2}{2c_1}};
\end{aligned}$$

$$\begin{aligned}
\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_4} &\simeq 0, \\
\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_4} &\simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\, \frac{1}{2c} w^2}, \\
\ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_4} &\simeq \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1}}.
\end{aligned}$$

The corresponding diagrams are shown on pp. 78–80.

The following lemma shows that the upper rows of the diagrams for $\widehat{\mathcal{S}}_2$, $\widehat{\mathcal{S}}_3$ and $\widehat{\mathcal{S}}_4$ are indeed exact. A choice of the matrix D amounts to a determination of the cokernel (which is unique up to unique isomorphism). In the above diagrams, we chose $D = \pm\mathbb{1}$ in such a way that we obtain nice representatives of the gluing matrices in the next section.

Lemma 3.19. *Let X be a topological space, and let $Z \subseteq X$ be a locally closed subset. Let $\varphi_j, \psi_j \colon X \to \mathbb{R}$, $j \in \{1, \dots, n\}$, be continuous functions such that $\varphi_1(x) > \varphi_2(x) > \dots > \varphi_n(x)$ and $\varphi_j(x) \geq \psi_j(x)$ for any $j \in \{1, \dots, n\}$ and any $x \in Z$. Let $A \in \mathbf{k}^{n\times n}$ be an invertible, lower triangular matrix. Let $D \in \mathbf{k}^{n\times n}$ be diagonal and invertible, and set $B := DA^{-1}$. Then the sequence in* $\mathrm{Mod}(\mathbf{k}_{X\times\mathbb{R}})$

$$0 \longrightarrow \bigoplus_{j=1}^{n} \mathsf{E}_{Z|X}^{\varphi_j \triangleright \psi_j} \xrightarrow{A} \bigoplus_{j=1}^{n} \mathsf{E}_{Z|X}^{\varphi_j} \xrightarrow{B} \bigoplus_{j=1}^{n} \mathsf{E}_{Z|X}^{\psi_j} \longrightarrow 0$$

is exact.

Proof. It suffices to show that the stalk of this sequence at any point of $Z \times \mathbb{R}$ is exact. (Note that the stalks at points of $(X \smallsetminus Z) \times \mathbb{R}$ are all zero.)

Let $(\check{x}, \check{t}) \in Z \times \mathbb{R}$, and let $l \in \{0, \dots, n\}$ be the unique number such that $-\varphi_j(\check{x}) \leq \check{t}$ for all $j \leq l$ and $\check{t} < -\varphi_j(\check{x})$ for all $j > l$. Then the stalk of the above sequence at $(\check{x}, \check{t})$ is

$$0 \longrightarrow \bigoplus_{j=1}^{n} \bigstar_j \xrightarrow{\widetilde{A}} \bigoplus_{j=1}^{n} \blacktriangle_j \xrightarrow{\widetilde{B}} \bigoplus_{j=1}^{n} \star_j \longrightarrow 0. \tag{3.24}$$

Here $\bigstar_j$, $\blacktriangle_j$ and $\star_j$ are the stalks of the direct summand of the sequence in question. We have $\bigstar_j = \blacktriangle_j = \star_j = 0$ for $j > l$ since in this case one has $\check{t} < -\varphi_j(\check{x})$, and in particular $\check{t} < -\psi_j(\check{x})$. For $j \leq l$, we get $\blacktriangle_j = \mathbf{k}$, so the object in the middle is nothing but $\mathbf{k}^l$ (regarded as the first l components of $\mathbf{k}^n$). Furthermore, still for $j \leq l$, we have $\bigstar_j, \star_j \in \{0, \mathbf{k}\}$, and $\bigstar_j = \mathbf{k}$ if and only if $\star_j = 0$.

The first map is given by applying the matrix A, whose image is in $\mathbf{k}^n$, and then projecting to the space $\mathbf{k}^l$ (the first l entries). Hence, the matrix $\widetilde{A}$ is obtained from A by replacing the entries of lines $l+1, \dots, n$ by zeros. Similarly, $\widetilde{B}$ is obtained from B by replacing the entries of the lines where $\star_j = 0$ by zeros.

In other words, the sequence (3.24) is isomorphic to

$$0 \longrightarrow \bigoplus_{j=1}^{l} \bigstar_j \xrightarrow{\widehat{A}} \mathbf{k}^l \xrightarrow{\widehat{B}} \bigoplus_{j=1}^{l} \star_j \longrightarrow 0.$$

Here, $\widehat{A}$ is the $l \times l$ matrix in the upper left corner of A. Moreover, we have $\widehat{B} = \widehat{D} \cdot \widehat{A^{-1}}$, where $\widehat{A^{-1}}$ is the $l \times l$ matrix in the upper left corner of A^{-1} and $\widehat{D}$ is obtained from D by taking the $l \times l$ matrix in the upper left corner of D and, for $1 \leq j \leq l$, replacing the jth diagonal entry by zero if $\star_j = 0$. It is an easy observation that $\widehat{A}\widehat{A^{-1}} = \mathbb{1}$, the $l \times l$ identity matrix, since A and A^{-1} are lower-triangular. In particular, $\widehat{A}$ is invertible and $\widehat{A}^{-1} = \widehat{A^{-1}}$. Injectivity of $\widehat{A}$ is therefore clear. Surjectivity of $\widehat{B}$ holds because it is the composition of an invertible map with a (surjective) projection. Finally, the kernel of $\widehat{D}$ is exactly $\bigoplus_{j=1}^{l} \bigstar_j$, so the kernel of $\widehat{B}$ is the preimage of $\bigoplus_{j=1}^{l} \bigstar_j$ under $\widehat{A^{-1}}$, which is the image of $\widehat{A}$. Therefore, (3.24) is exact, as was to be shown. □

Diagram for $\widehat{\mathcal{S}}_2$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c} w^2} \arrow[r, "\sigma_2"] \arrow[d, "(\sigma_1^{-1}, \mathbb{1})"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1}} \arrow[r, "\sigma_2^{-1}"] \arrow[d, "(\sigma_4\sigma_3, \mathbb{1})"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\mathrm{Re}\,\frac{1}{2c} w^2} \arrow[r] & 0 \\
 & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0 \rhd \eta_c} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c} w^2} \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[d, "\sigma_1 - \mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1}} \arrow[d, "\mathbb{1} - \sigma_4\sigma_3"] & & \\
 & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0 \rhd \eta_c} \arrow[r, "\sigma_1^{-1}"] \arrow[d] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Diagram for $\widehat{\mathcal{S}}_3$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus\limits_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\, \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_3} \arrow[r, "-\mathbb{1}"] \arrow[d, "(\sigma_3{,}\mathbb{1})"] & \bigoplus\limits_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \arrow[r, "\mathbb{1}"] \arrow[d, "(\sigma_4\sigma_3{,}\mathbb{1})"] & \bigoplus\limits_{c\in C} \mathsf{E}^{\mathrm{Re}\, \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_3} \arrow[r] & 0 \\
 & \bigoplus\limits_{c\in C} \mathsf{E}^{0 \triangleright \eta_c}_{\widehat{\mathcal{S}}_3} \oplus \bigoplus\limits_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\, \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_3} \arrow[r, "-(\sigma_4|\mathbb{1})"] \arrow[d, "\mathbb{1}-\sigma_3"] & \bigoplus\limits_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \oplus \bigoplus\limits_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \arrow[d, "\mathbb{1}-\sigma_4\sigma_3"] & & \\
 & \bigoplus\limits_{c\in C} \mathsf{E}^{0 \triangleright \eta_c}_{\widehat{\mathcal{S}}_3} \arrow[r, "-\sigma_4"] \arrow[d] & \bigoplus\limits_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Diagram for $\widehat{\mathcal{S}}_4$:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \arrow[r, "-\sigma_4"] \arrow[d, "(\mathbb{1}{,}\sigma_3^{-1})"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1}} \arrow[r, "\sigma_4^{-1}"] \arrow[d, "(\mathbb{1}{,}\sigma_2\sigma_1)"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\operatorname{Re} \frac{1}{2c} w^2} \arrow[r] & 0 \\
 & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \oplus \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0 \triangleright \eta_c} \arrow[r, "-(\sigma_4|\mathbb{1})"] \arrow[d, "\mathbb{1}-\sigma_3"] & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1}} \oplus \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0} \arrow[d, "\mathbb{1}-\sigma_4\sigma_3"] & & \\
 & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0 \triangleright \eta_c} \arrow[r, "-\sigma_4"] \arrow[d] & \bigoplus\limits_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Gluing matrices

We have seen in Proposition 3.18 that ${}^{\mathcal{L}}\mathcal{F}$ is isomorphic to a direct sum of exponential enhanced sheaves on each of the $\widehat{\mathcal{S}}_k$ (and such isomorphisms have actually been constructed). Therefore, on each of the half-lines $\widehat{\mathcal{S}}_{k,k+1}$, we have two trivializing isomorphisms $\widehat{\alpha}_k$ and $\widehat{\alpha}_{k+1}$ coming from the ones on the two adjacent sectors. Our aim is to find matrices $\widehat{\sigma}_k$ representing automorphisms of $\bigoplus_{c\in C} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{k,k+1}}$ such that the following diagram commutes for any $k \in \mathbb{Z}/4\mathbb{Z}$:

$$\begin{tikzcd}
\bigoplus_{c\in C} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{k,k+1}} \arrow[rr, "\widehat{\sigma}_k"] & & \bigoplus_{c\in C} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{k,k+1}} \\
& ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_{k,k+1}} \arrow[ul, "\simeq", "\widehat{\alpha}_k"'] \arrow[ur, "\simeq"', "\widehat{\alpha}_{k+1}"] &
\end{tikzcd}$$

We obtain the following result.

Proposition 3.20. *Gluing matrices for ${}^{\mathcal{L}}\mathcal{F}$ are given by $\widehat{\sigma}_k = \sigma_k$, $k \in \mathbb{Z}/4\mathbb{Z}$.*

Proof. In order to prove the assertion, we need to make the isomorphisms constructed in the proof of the previous proposition more explicit. Let us give the proof for $\widehat{\sigma}_1 = \sigma_1$.

By what we have learnt in Proposition 3.18, the triangle (3.17) is actually a short exact sequence (identifying ${}^{\mathcal{L}}\mathcal{F}$ with $\mathrm{H}^0({}^{\mathcal{L}}\mathcal{F})$)

$$0 \longrightarrow \ker(\sigma_1 - \mathbb{1}) \oplus \ker(\mathbb{1} - \sigma_3) \xrightarrow{(\mathbb{1}|\sigma_2)-(\sigma_4|\mathbb{1})} \ker(\mathbb{1} - \sigma_4\sigma_3) \longrightarrow {}^{\mathcal{L}}\mathcal{F} \longrightarrow 0. \tag{3.25}$$

On $\widehat{\mathcal{S}}_1$ (i.e. applying $(\bullet)_{\widehat{\mathcal{S}}_1}$), it induces

$$0 \longrightarrow \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \xrightarrow{\mathbb{1}|\sigma_2} \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \longrightarrow ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1} \longrightarrow 0, \tag{3.26}$$

since we proved $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1} \simeq 0$. Moreover, we obtained determinations of $\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1}$ and $\ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1}$ from the isomorphisms of short exact sequences given by

$$\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \oplus \bigoplus_{c\in C} \mathsf{E}^{0 \rhd \eta_c}_{\widehat{\mathcal{S}}_1} \arrow[r, "\sigma_1 - \mathbb{1}"] \arrow[d, equal] & \bigoplus_{c\in C} \mathsf{E}^{0 \rhd \eta_c}_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \arrow[r, "(\mathbb{1}{,}\sigma_1)"] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \oplus \bigoplus_{c\in C} \mathsf{E}^{0 \rhd \eta_c}_{\widehat{\mathcal{S}}_1} \arrow[r, "\sigma_1 - \mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}^{0 \rhd \eta_c}_{\widehat{\mathcal{S}}_1} \arrow[r] & 0
\end{tikzcd} \tag{3.27}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \oplus \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] \arrow[d, equal] & \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \arrow[r, "(\mathbb{1},\sigma_2\sigma_1)"] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \oplus \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \arrow[r] & 0.
\end{tikzcd} \tag{3.28}$$

Combining the red and green isomorphisms with (3.26), we obtained the isomorphism (in blue) whose existence was claimed in Proposition 3.18:

$$\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_1} \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[d, "\simeq"] & \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \arrow[r, "\mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \arrow[r, "\mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}^{\operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \arrow[r] & 0.
\end{tikzcd} \tag{3.29}$$

The same principle applies to the proof for $\widehat{\mathcal{S}}_2$: We start from the sequence

$$0 \longrightarrow \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_2} \xrightarrow{\mathbb{1}|\sigma_2} \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2} \longrightarrow ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_2} \longrightarrow 0, \tag{3.30}$$

which is obtained by restricting (3.25) to $\widehat{\mathcal{S}}_2$ and remembering that $\ker(\mathbb{1}-\sigma_3)_{\widehat{\mathcal{S}}_2} \simeq 0$. The determinations of the kernels are given by the diagrams (we use the same colors, but dashed arrows)

$$\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, dashed, "\simeq"] & \bigoplus_{c\in C} \mathsf{E}^{0\triangleright\eta_c}_{\widehat{\mathcal{S}}_2} \oplus \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \arrow[r, "\sigma_1-\mathbb{1}"] \arrow[d, equal] & \bigoplus_{c\in C} \mathsf{E}^{0\triangleright\eta_c}_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \arrow[r, "(\sigma_1^{-1},\mathbb{1})"] & \bigoplus_{c\in C} \mathsf{E}^{0\triangleright\eta_c}_{\widehat{\mathcal{S}}_2} \oplus \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \arrow[r, "\sigma_1-\mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}^{0\triangleright\eta_c}_{\widehat{\mathcal{S}}_2} \arrow[r] & 0
\end{tikzcd} \tag{3.31}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, dashed, "\simeq"] & \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \oplus \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] \arrow[d, equal] & \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \arrow[r, "(\sigma_4\sigma_3,\mathbb{1})"] & \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \oplus \bigoplus_{c\in C} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \bigoplus_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \arrow[r] & 0,
\end{tikzcd} \tag{3.32}$$

and one combines the red isomorphisms with (3.30) to obtain

$$\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_2} \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[d, dashed, "\simeq"] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, dashed, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \arrow[r, "\sigma_2"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1}} \arrow[r, "\sigma_2^{-1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\operatorname{Re} \frac{1}{2c} w^2} \arrow[r] & 0.
\end{tikzcd} \tag{3.33}$$

Restricting the diagrams (3.29) and (3.33) to the half-line $\widehat{\mathcal{S}}_{12}$ (more precisely, applying the functor $(\bullet)_{\widehat{\mathcal{S}}_{12}}$), one obtains two diagrams whose first rows are identical, since they are originally induced by (3.25). Hence, one can put them together into one diagram as follows (note that $w_1 = 0$ on $\widehat{\mathcal{S}}_{12}$):

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0 \triangleright \operatorname{Re} \frac{1}{2c} w^2} \arrow[r, "\mathbb{1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "\mathbb{1}"] \arrow[dd, bend left] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{\operatorname{Re} \frac{1}{2c} w^2} \arrow[r] \arrow[dd, bend left, "\widehat{\sigma}_1"] & 0 \\
0 \arrow[r] & \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_{12}} \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_{12}} \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_{12}} \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0 \triangleright \operatorname{Re} \frac{1}{2c} w^2} \arrow[r, "\sigma_2"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "\sigma_2^{-1}"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{\operatorname{Re} \frac{1}{2c} w^2} \arrow[r] & 0.
\end{tikzcd} \tag{3.34}$$

Now note that the purple morphism on the right is the gluing map between $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ which we are interested in, so it is given by the matrix $\widehat{\sigma}_1$. Therefore, in order to find this matrix, one needs to determine the composition of the dashed green isomorphism with the inverse of the solid green isomorphism, i.e. the orange morphism shown in the diagram.

In order to find the orange arrow, we apply the functor $(\bullet)_{\widehat{\mathcal{S}}_{12}}$ to the diagrams (3.28) and (3.32) and put them together to obtain (note that $w_1 = 0$ on $\widehat{\mathcal{S}}_{12}$)

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "(\mathbb{1}{,}\sigma_2\sigma_1)"] \arrow[dd, bend left] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r] & 0 \\
0 \arrow[r] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_{12}} \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] \arrow[u, equal] \arrow[d, equal] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "(\sigma_4\sigma_3{,}\mathbb{1})"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \oplus \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \bigoplus_{c\in C} \mathsf{E}_{\widehat{\mathcal{S}}_{12}}^{0} \arrow[r] & 0.
\end{tikzcd}$$

It is easy to see that the orange arrow must be given by the matrix $\sigma_2\sigma_1$ to make the diagram commute (recall that $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$).

Finally, the right square of (3.34) reads as (omitting the middle row)

$$\begin{tikzcd} \bigoplus\limits_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{12}} \arrow[r, "\mathbb{1}"] \arrow[d, "\sigma_2\sigma_1"] & \bigoplus\limits_{c\in C} \mathsf{E}^{\operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{12}} \arrow[d, "\widehat{\sigma}_1"] \\ \bigoplus\limits_{c\in C} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{12}} \arrow[r, "\sigma_2^{-1}"] & \bigoplus\limits_{c\in C} \mathsf{E}^{\operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{12}}, \end{tikzcd}$$

and therefore $\widehat{\sigma}_1 = \sigma_1$. The proofs of the other cases are completely analogous. □

Remark.

- Although the proofs of the other cases are analogous, one might be interested in the case $\widehat{\sigma}_2 = \sigma_2$ (the transition from $\widehat{\mathcal{S}}_2$ to $\widehat{\mathcal{S}}_3$): At first glance the situation seems to be different since $\ker(\mathbb{1}-\sigma_3)_{\widehat{\mathcal{S}}_2} \simeq 0$ but $\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_3} \simeq 0$. However, this means that $\ker(\mathbb{1}-\sigma_3)_{\widehat{\mathcal{S}}_{23}} \oplus \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_{23}} \simeq 0$, so in the combined sequence on $\widehat{\mathcal{S}}_{23}$ the first object just vanishes. This also reflects the fact that $w_2 = \frac{c_2}{c_1}w_1$ and hence $\frac{w_1^2}{2c_1} = \operatorname{Re}\frac{1}{2c}w^2$ on $\widehat{\mathcal{S}}_{23}$.
- In the proof of Proposition 3.20, we actually construct an isomorphism between the diagrams for $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ (both restricted to $\widehat{\mathcal{S}}_{12}$). The starting point is thereby the identification of the lower parts of the two diagrams by identities, which expresses the fact that both diagrams have a common "skeleton". All the other isomorphisms are then induced by these identities. They are, however, not identities any more since the realizations of kernels and cokernels differ on $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ (cf. (3.21)).

Pure Gaussian type of ${}^{\mathcal{L}}\mathcal{F}$

To finish the proof of Theorem 3.7, it remains to conclude that ${}^{\mathcal{L}}\mathcal{F}$ is again of pure Gaussian type. We resume writing the ranks r_c and the second index in the exponentials from now on.

Up to now, we have shown that (see Proposition 3.18)

$$({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_k} \simeq \bigoplus_{c\in C} \left(\mathsf{E}^{\operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_k|\mathbb{C}_w}\right)^{r_c} \tag{3.35}$$

for $k \in \mathbb{Z}/4\mathbb{Z}$, and the gluing matrices are $\sigma_1,\dots,\sigma_4$. Setting $\widehat{C} := -1/C = \{-\frac{1}{c} \mid c \in C\}$ and $\widehat{r}_{\widehat{c}} := r_{-1/\widehat{c}}$, we can write

$$({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_k} \simeq \bigoplus_{\widehat{c}\in \widehat{C}} \left(\mathsf{E}^{-\operatorname{Re}\frac{\widehat{c}}{2}w^2}_{\widehat{\mathcal{S}}_k|\mathbb{C}_w}\right)^{r_{\widehat{c}}}. \tag{3.36}$$

The last step before making the connection to Definition 2.14 is to replace the sectors $\widehat{\mathcal{S}}_k$ by right-angled sectors. One has $\arg \widehat{C} = \pi - \arg C \in (\frac{\pi}{2}, \frac{3\pi}{2})$ if $\arg C \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so the Stokes directions are $\frac{\pi}{4} - \frac{1}{2}\arg \widehat{C} + k\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{2}\arg C + k\frac{\pi}{2}$, $k \in \mathbb{Z}/4\mathbb{Z}$. Clearly, one can also write the Stokes directions in the form

$$\widehat{\mathrm{st}}_k = \frac{3\pi}{4} + \frac{1}{2}\arg C + k\frac{\pi}{2},$$

which only amounts to a change of numbering. It is easy to check that $\widehat{\mathrm{st}}_k$ is the axis direction of $\widehat{\mathcal{S}}_k$ for any $k \in \mathbb{Z}/4\mathbb{Z}$ (see Fig. 3.6). Therefore, each of the sectors $\mathcal{S}_k$ contains exactly one Stokes direction. Moreover, $\widehat{\theta}_0 := \pi - \theta_0 = \pi + \frac{1}{2}\arg C$ is a generic direction for $\widehat{C}$, and the numbering on $\widehat{C}$ with respect to $\widehat{\theta}_0$ is in fact the one inherited from the ordering on C with respect to θ_0, i.e. $\widehat{c}_{(1)} <_{\widehat{\theta}_0} \widehat{c}_{(2)} <_{\widehat{\theta}_0} \dots <_{\widehat{\theta}_0} \widehat{c}_{(n)}$ if $c_{(1)} <_{\theta_0} c_{(2)} <_{\theta_0} \dots <_{\theta_0} c_{(n)}$. This follows from the easily verified equivalences

$$\begin{aligned}
&c <_{\theta_0} d \\
&\iff \operatorname{Re} \frac{c}{2} z^2 < \operatorname{Re} \frac{d}{2} z^2 \text{ for } z \in \mathbb{C} \text{ with } z \neq 0 \text{ and } \arg z = \theta_0 = -\frac{1}{2}\arg C \\
&\iff |c| < |d| \\
&\iff -\operatorname{Re} \frac{1}{2c} w^2 < -\operatorname{Re} \frac{1}{2d} w^2 \text{ for } w \in \mathbb{C} \text{ with } w \neq 0 \text{ and } \arg w = \widehat{\theta}_0 = \pi + \frac{1}{2}\arg C \\
&\iff \operatorname{Re} \frac{\widehat{c}}{2} w^2 < \operatorname{Re} \frac{\widehat{d}}{2} w^2 \text{ for } w \in \mathbb{C} \text{ with } w \neq 0 \text{ and } \arg w = \widehat{\theta}_0 = \frac{3\pi}{2} - \frac{1}{2}\arg \widehat{C} \\
&\iff \widehat{c} <_{\widehat{\theta}_0} \widehat{d}
\end{aligned}$$

for $c, d \in C$, i.e. $\arg c = \arg d = \arg C$. This observation is important because the representation of the gluing isomorphisms by matrices depends on the numbering of the direct sum. Hence, the choice of such a suitable generic direction $\widehat{\theta}_0$ ensures that the gluing matrices induced by (3.36) are in fact the gluing matrices $\widehat{\sigma}_k = \sigma_k$ coming from (3.35), and there is not an interchange of rows or columns.

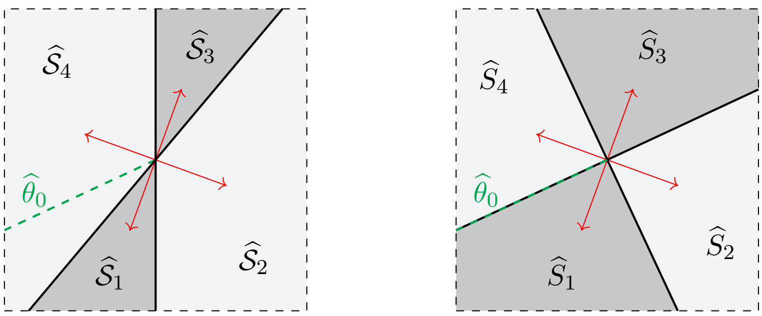


Figure 3.6.: Each of the sectors $\widehat{\mathcal{S}}_k$ (on the left) contains exactly one Stokes direction. Consequently, the sectors $\widehat{\mathcal{S}}_k$ can be deformed to the sectors $\widehat{S}_k$ (on the right) without crossing a Stokes line.

Moreover, it is easy to see that $\widehat{\theta}_0$ lies between st_0 and st_1. Therefore, one can use the sectors $\widehat{S}_k := \{w \in \mathbb{C} \mid \arg w \in [\widehat{\theta}_0 + (k-1)\frac{\pi}{2}, \widehat{\theta}_0 + k\frac{\pi}{2}] \text{ if } w \neq 0\}$ instead of $\widehat{\mathcal{S}}_k$, since by Lemma 3.9 one will still have trivializations

$$({}^{\mathcal{L}}\mathcal{F})_{\widehat{S}_k} \simeq \bigoplus_{\widehat{c} \in \widehat{C}} (\mathsf{E}^{-\operatorname{Re}\frac{\widehat{c}}{2}w^2}_{\widehat{S}_k|\mathbb{C}_w})^{r_{\widehat{c}}}$$

and gluing matrices $\sigma_1, \dots, \sigma_4$.

In conclusion, we obtain ${}^{\mathcal{L}}\mathcal{F} \simeq \mathcal{F}_\sigma^{\widehat{C}, \widehat{\theta}_0, \widehat{\mathfrak{r}}}$, since such a sheaf is unique up to unique isomorphism by Lemma A.6. This completes the proof of Theorem 3.7.

3.4. A more general case

In this section, we will finally consider a case where the parameters are not aligned along a half-line through the origin as in Section 3.3. We will work with only two parameters here, so $C = \{c, d\} \subset \mathbb{C}^\times$, and we will assume $r_c = r_d = 1$. (As we have seen in the previous section, this last assumption does not change the proof, but only makes it more readable.) We will still impose conditions on the parameters which will ensure that they are not "too far apart". This will be made explicit below, and is not related to the absolute value of the parameters' difference, but rather to its argument. It turns out that the methods of the previous section can be adapted to this situation. As in the previous sections, the key to computing the Fourier–Laplace transform of a D-module of pure Gaussian type is the corresponding computation for the associated enhanced sheaves of pure Gaussian type.

3.4.1. Assumptions and main statement

Let $C = \{c, d\} \subset \mathbb{C}^\times$ be a set of two parameters and let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ be of pure Gaussian type C with ranks of the regular parts in the Levelt–Turrittin decomposition given by $r_c = r_d = 1$.

In the case in which the parameters' arguments coincided and $-\frac{\pi}{2} < \arg C < \frac{\pi}{2}$, we used sectors $\mathcal{S}_k$ whose boundaries were the half-lines defined by the angles 0, $\frac{\pi}{2} - \arg C$, π and $-\frac{\pi}{2} - \arg C$. These sectors were chosen in such a way that, after transforming them into new coordinates in order to write the hyperbola in standard form, they became right-angled sectors with horizontal and vertical boundaries.

We would like to use these sectors (for $\arg c$ instead of $\arg C$) also here in order to apply the results found in Section 3.3.3. If $c \in \mathbb{C}^\times$ and $d = c + \rho \cdot e^{i\omega}$ for some $\rho \in \mathbb{R}_{>0}$ and $\omega \in \mathbb{R}/2\pi\mathbb{Z}$, the Stokes directions are the values of $\arg z$ solving equation (2.1), which in this case is equivalent to

$$\rho \cos\omega \cos(2\arg z) - \rho \sin\omega \sin(2\arg z) = 0.$$

By a standard identity of trigonometry, this is again equivalent to

$$\rho \cos(\omega + 2\arg z) = 0,$$

so the Stokes directions are

$$\mathrm{st}_k = -\frac{\pi}{4} - \frac{\omega}{2} + k\frac{\pi}{2}, \quad k \in \mathbb{Z}/4\mathbb{Z}.$$

Note that this involves a determination of ω, which, however, does not change the set of all four Stokes directions. Here, we choose $\frac{\omega}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2})$.

Let us assume that $0 \leq \arg c < \frac{\pi}{2}$. In order to ensure that each of the sectors contains exactly one Stokes direction, we could ask that $-\frac{\pi}{2} + 2\arg c < \omega < \frac{\pi}{2}$. This guarantees that the Stokes direction st_1 is contained in $\mathcal{S}_1$ (and is not its boundary). Consequently, since the central angle of $\mathcal{S}_1$ is acute, each of the sectors contains exactly one Stokes direction. In the course of the proof, it will turn out to be useful to know the relation between the arguments of the two parameters, so let us moreover assume that $\arg d \geq \arg c$.

To sum up, we consider the case where $\arg c \in [0, \frac{\pi}{2})$ and $\omega \in [\arg c, \frac{\pi}{2})$. We formulate the assumptions in the following equivalent manner, illustrated in Fig. 3.7.

Condition 3.21. We say that an ordered pair (c, d) of nonzero complex numbers $c, d \in \mathbb{C}^\times$ satisfies condition ($\llcorner$) if the following is satisfied:

$$c_1 > 0, \quad c_2 \geq 0, \quad d_1 > c_1 \quad \text{and} \quad \frac{d_2}{d_1} \geq \frac{c_2}{c_1}, \tag{$\llcorner$}$$

where we write $c = c_1 + ic_2$ and $d = d_1 + id_2$ with their real and imaginary parts. Note that, in particular, this also implies $d_2 > c_2$. The last condition can be reformulated as $c_1 d_2 - c_2 d_1 \geq 0$.

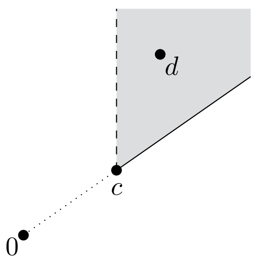


Figure 3.7.: Let $\arg c \in [0, \frac{\pi}{2})$. A pair (c, d) satisfies condition ($\llcorner$) if and only if d lies in the cone with vertex c and bounded by the directions $\arg c$ (included) and $\frac{\pi}{2}$ (excluded).

It is not difficult to see that, under this assumption, $\theta_0 = -\frac{1}{2}\arg c$ is a generic direction: We have $\mathrm{st}_0 \in (-\frac{\pi}{2}, -\frac{\pi}{4} - \frac{1}{2}\arg c]$ and $\mathrm{st}_1 \in (0, \frac{\pi}{2} - \frac{1}{2}\arg c]$, but clearly $\theta_0 \in (-\frac{\pi}{4} - \frac{1}{2}\arg c, 0]$.

We will write $\mathcal{F}_\sigma^{C,\theta_0}$ instead of $\mathcal{F}_\sigma^{C,\theta_0,\mathbb{r}}$ since all the ranks are assumed to be 1. In this situation, we have the following theorem, similar to Theorem 3.7 in the aligned case.

Theorem 3.22. *Let $C = \{c, d\} \subset \mathbb{C}^\times$, and assume that (c, d) satisfies condition ($\llcorner$). Set $\theta_0 := -\frac{1}{2}\arg c$. Let $\sigma = (\sigma_k)_{k \in \mathbb{Z}/4\mathbb{Z}}$ be a family of four invertible matrices $\sigma_k \in \mathbf{k}^{2\times 2}$ such that σ_1 and σ_3 are upper-triangular, σ_2 and σ_4 are lower-triangular and $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathbb{1}$.*

We define $\widehat{C} := \{-\frac{1}{c}, -\frac{1}{d}\}$, $\widehat{\theta}_0 := \pi - \theta_0$. *Then there is an isomorphism*

$$^{\mathcal{L}}\mathcal{F}_\sigma^{C,\theta_0} \simeq \mathcal{F}_\sigma^{\widehat{C},\widehat{\theta}_0}.$$

In particular, the gluing matrices $\sigma = (\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ *remain the same.*

As an immediate consequence, we obtain the following corollary.

Corollary 3.23. *Let* $C = \{c,d\} \subset \mathbb{C}^\times$ *such that* (c,d) *satisfies condition* ($\llcorner$), *and let* $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}})$ *be of pure Gaussian type* C. *Let moreover* $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ *be Stokes multipliers for* $\mathcal{M}$ *with respect to the generic direction* $\theta_0 = -\frac{1}{2}\arg c$. *Then the Fourier–Laplace transform* $^{\mathsf{L}}\mathcal{M}$ *of* $\mathcal{M}$ *is of pure Gaussian type* $\widehat{C} = \{-\frac{1}{c}, -\frac{1}{d}\}$, *and a family of Stokes multipliers of* $^{\mathsf{L}}\mathcal{M}$ *with respect to the generic direction* $\widehat{\theta}_0$ *is given by* $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$.

In the remainder of this section, we give a proof of Theorem 3.22.

3.4.2. Enhanced Fourier–Sato transform of exponentials

First, we choose a covering of the complex plane by the closed sectors (as in Section 3.3.2, with $\arg C$ replaced by $\arg c$)

$$\begin{aligned}
\mathcal{S}_1 &:= \left\{ z \in \mathbb{C}_z \;\middle|\; \arg z \in \left[0, \frac{\pi}{2} - \arg c\right] \text{ if } z \neq 0 \right\}, \\
\mathcal{S}_2 &:= \left\{ z \in \mathbb{C}_z \;\middle|\; \arg z \in \left[\frac{\pi}{2} - \arg c, \pi\right] \text{ if } z \neq 0 \right\}, \\
\mathcal{S}_3 &:= \left\{ z \in \mathbb{C}_z \;\middle|\; \arg z \in \left[-\pi, -\frac{\pi}{2} - \arg c\right] \text{ if } z \neq 0 \right\}, \\
\mathcal{S}_4 &:= \left\{ z \in \mathbb{C}_z \;\middle|\; \arg z \in \left[-\frac{\pi}{2} - \arg c, 0\right] \text{ if } z \neq 0 \right\}.
\end{aligned}$$

As discussed above, the assumptions on C ensure that each of these sectors contains one and only one Stokes direction. Moreover, θ_0 and the horizontal border of $\mathcal{S}_1$ lie between the same Stokes directions. Therefore, we can replace the standard rectangular sectors S_k by the $\mathcal{S}_k$ in view of Lemma 3.9, i.e. $\mathcal{F}_\sigma^{C,\theta_0}$ will still decompose as a direct sum of two exponentials on these sectors (with gluing matrices σ_k).

The exponential enhanced sheaves involved in the present case are thus $\mathsf{E}_{\mathcal{S}_k|\mathbb{C}}^{-\operatorname{Re}\frac{c}{2}z^2}$ and $\mathsf{E}_{\mathcal{S}_k|\mathbb{C}}^{-\operatorname{Re}\frac{d}{2}z^2}$ for $k \in \mathbb{Z}/4\mathbb{Z}$. The enhanced Fourier–Sato transform of the first type of exponential has been calculated in Section 3.3.3. We will not write the index c here, i.e. we will write η, φ_{r}^+ etc. for η_c, $\varphi_{\mathrm{r},c}^+$ etc. in the sequel.

The aim of this section is to find the enhanced Fourier–Sato transform of the exponentials with exponent $-\operatorname{Re}\frac{d}{2}z^2$. This is not covered by Section 3.3.3 if $\arg d \neq \arg c$. However, it works along the same lines:

By (3.6), the space to be studied in order to determine the stalk of the cohomology sheaves of $^{\mathcal{L}}\mathsf{E}_{\mathcal{S}_k|\mathbb{C}_z}^{-\operatorname{Re}\frac{d}{2}z^2}$ at a point $(\check{w},\check{t}) \in \mathbb{C}_w \times \mathbb{R}$ is the intersection of the hyperbolic region defined by

$$\check{t} - \operatorname{Re}\left(z\check{w} + \frac{c}{2}z^2\right) \geq 0$$

and the sector $\mathcal{S}_k$. Let us explicitly treat the case $k=1$, since the others are analogous.

We change coordinates in order to get the hyperbola in canonical form, setting $x_1 := z_1 - \frac{d_2}{d_1}z_2 + \frac{\breve{w}_1}{d_1}$ and $x_2 := z_2 + \frac{d_1\breve{w}_2 - d_2\breve{w}_1}{|d|^2}$. The hyperbolic region is then given by

$$\frac{d_1}{2}x_1^2 - \frac{|d|^2}{2d_1}x_2^2 \leq \breve{t} + \operatorname{Re}\frac{1}{2d}\breve{w}^2. \tag{3.37}$$

The sector $\mathcal{S}_1$ is transformed into the sector bounded by the half-lines with directions $(1,0)^{\mathrm{T}}$ and $(c_2d_1 - c_1d_2, c_1d_1)^{\mathrm{T}}$ and centered at the point $\left(\frac{\breve{w}_1}{d_1}, \frac{d_1\breve{w}_2 - d_2\breve{w}_1}{|d|^2}\right)$, i.e. the sector given by the inequalities

$$(d_2c_1 - c_2d_1)\left(x_2 - \frac{d_1\breve{w}_2 - d_2\breve{w}_1}{|d|^2}\right) \geq -c_1d_1\left(x_1 - \frac{\breve{w}_1}{d_1}\right), \qquad x_2 \geq \frac{d_1\breve{w}_2 - d_2\breve{w}_1}{|d|^2}. \tag{3.38}$$

In Section B.3.1, we describe the conditions under which the intersection of the hyperbolic region (3.37) with the sector (3.38) has non-vanishing compactly supported cohomology. The additional difficulty here is that the sector is not right-angled as in the aligned case any more. By (3.6), this allows us to conclude that

$$\mathrm{H}^l\big({}^{\mathcal{L}}\mathsf{E}_{\mathcal{S}_1|\mathbb{C}_z}^{-\operatorname{Re}\frac{d}{2}z^2}\big) \simeq 0 \text{ for } l \neq -1$$

and

$$\mathrm{H}^{-1}\big({}^{\mathcal{L}}\mathsf{E}_{\mathcal{S}_1|\mathbb{C}_z}^{-\operatorname{Re}\frac{d}{2}z^2}\big)_{(\breve{w},\breve{t})} \simeq \begin{cases} \mathbf{k} & \text{if } w_2 \leq \frac{c_2}{c_1}w_1,\ w_2 \leq \frac{d_2}{d_1}w_1 \text{ and } -\psi_{\mathrm{r}}^+(\breve{w}) \leq \breve{t} < -\psi_{\mathrm{r}}^-(\breve{w}), \\ 0 & \text{otherwise} \end{cases}$$

with the continuous functions $\psi_{\mathrm{r}}^+, \psi_{\mathrm{r}}^- \colon \mathbb{C}_w \to \mathbb{R}$ defined by

$$\psi_{\mathrm{r}}^+(w) := \begin{cases} \frac{w_1^2}{2d_1} & \text{if } w_1 \leq 0, \\ 0 & \text{if } w_1 > 0 \end{cases}$$

and

$$\psi_{\mathrm{r}}^-(w) := \begin{cases} \operatorname{Re}\frac{1}{2d}w^2 & \text{if } (c_1d_2 - c_2d_1)w_2 \leq -(c_1d_1 + c_2d_2)w_1, \\ -\frac{(c_1w_2 - c_2w_1)^2}{2(c_1^2d_1 - c_2^2d_1 + 2c_1c_2d_2)} =: \zeta(w) & \text{if } (c_1d_2 - c_2d_1)w_2 > -(c_1d_1 + c_2d_2)w_1. \end{cases}$$

Setting

$$Y_1 := \left\{ w \in \mathbb{C}_w \;\middle|\; w_2 \leq \min\Big(\frac{c_2}{c_1}w_1, \frac{d_2}{d_1}w_1\Big) \right\},$$

the above observations on the stalks suggest an isomorphism ${}^{\mathcal{L}}\mathsf{E}_{\mathcal{S}_1|\mathbb{C}_z}^{-\operatorname{Re}\frac{d}{2}z^2} \simeq \mathsf{E}_{Y_1|\mathbb{C}_w}^{\psi_{\mathrm{r}}^+ \rhd \psi_{\mathrm{r}}^-}[1]$, whose proof is analogous to that of Proposition 3.10.

Similarly, we can proceed with the sectors $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$. Define the continuous functions

$\psi_1^+, \psi_1^- \colon \mathbb{C}_w \to \mathbb{R}$ by

$$\psi_1^+(w) := \begin{cases} 0 & \text{if } w_1 \le 0, \\ \frac{w_1^2}{2d_1} & \text{if } w_1 > 0 \end{cases}$$

and

$$\psi_1^-(w) := \begin{cases} \zeta(w) & \text{if } (c_1d_2 - c_2d_1)w_2 \le -(c_1d_1 + c_2d_2)w_1, \\ \operatorname{Re} \frac{1}{2d} w^2 & \text{if } (c_1d_2 - c_2d_1)w_2 > -(c_1d_1 + c_2d_2)w_1, \end{cases}$$

and set

$$Y_2 := \left\{ w \in \mathbb{C}_w \;\middle|\; w_2 \le \max\left(\frac{c_2}{c_1}w_1, \frac{d_2}{d_1}w_1\right) \right\},$$
$$Y_3 := \left\{ w \in \mathbb{C}_w \;\middle|\; w_2 \ge \max\left(\frac{c_2}{c_1}w_1, \frac{d_2}{d_1}w_1\right) \right\},$$
$$Y_4 := \left\{ w \in \mathbb{C}_w \;\middle|\; w_2 \ge \min\left(\frac{c_2}{c_1}w_1, \frac{d_2}{d_1}w_1\right) \right\}.$$

The regions Y_k are shown in Fig. 3.8 (p. 97). The global statements for all sectors are as follows.

Proposition 3.24. *There are isomorphisms in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$

$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_1|\mathbb{C}_z} \simeq \mathsf{E}^{\psi_{\mathrm{r}}^+ \rhd \psi_{\mathrm{r}}^-}_{Y_1|\mathbb{C}_w}[1],$$
$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_2|\mathbb{C}_z} \simeq \mathsf{E}^{\psi_1^+ \rhd \psi_1^-}_{Y_2|\mathbb{C}_w}[1],$$
$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_3|\mathbb{C}_z} \simeq \mathsf{E}^{\psi_1^+ \rhd \psi_1^-}_{Y_3|\mathbb{C}_w}[1],$$
$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_4|\mathbb{C}_z} \simeq \mathsf{E}^{\psi_{\mathrm{r}}^+ \rhd \psi_{\mathrm{r}}^-}_{Y_4|\mathbb{C}_w}[1].$$

Similar considerations can be made for the exponentials on the half-lines $\mathcal{S}_{k,k+1} := \mathcal{S}_k \cap \mathcal{S}_{k+1}$ for $k \in \mathbb{Z}/4\mathbb{Z}$ (see Section B.3.2). The result is the following.

Proposition 3.25. *There are isomorphisms in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$

$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{12}|\mathbb{C}_z} \simeq \mathsf{E}^{0 \rhd \zeta}_{\widehat{\mathcal{H}}_-|\mathbb{C}_w}[1],$$
$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{23}|\mathbb{C}_z} \simeq \mathsf{E}^{\psi_1^+}_{\mathbb{C}_w|\mathbb{C}_w}[1],$$
$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{34}|\mathbb{C}_z} \simeq \mathsf{E}^{0 \rhd \zeta}_{\widehat{\mathcal{H}}_+|\mathbb{C}_w}[1],$$
$$^{\mathcal{L}}\mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{41}|\mathbb{C}_z} \simeq \mathsf{E}^{\psi_{\mathrm{r}}^+}_{\mathbb{C}_w|\mathbb{C}_w}[1],$$

where $\widehat{\mathcal{H}}_- := \{w \in \mathbb{C}_w \mid c_1w_2 - c_2w_1 \le 0\}$ *and* $\widehat{\mathcal{H}}_+ := \{w \in \mathbb{C}_w \mid c_1w_2 - c_2w_1 \ge 0\}$.

Remark. Propositions 3.24 and 3.25 actually include the results of Section 3.3.3 as a special case: If $c_1d_2 - c_2d_1 = 0$, we have $Y_1 = Y_2 = \widehat{\mathcal{H}}_-$, $Y_3 = Y_4 = \widehat{\mathcal{H}}_+$, $\zeta = \eta_d$, $\psi_{\mathrm{r}}^+ = \varphi_{\mathrm{r},d}^+$, etc.

3.4.3. Enhanced Fourier–Sato transform of Gaussian enhanced sheaves

Let $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}})$ be of pure Gaussian type $C = \{c, d\}$. More precisely, let $\mathcal{F} := \mathcal{F}_\sigma^{C,\theta_0}$ for $\theta_0 = -\frac{1}{2}\arg C$ and some family of matrices $(\sigma_k)_{k\in\mathbb{Z}/4\mathbb{Z}}$ as in the assumptions of Theorem 3.22. Recalling Lemma 3.9, we have isomorphisms in $\mathrm{D}^\mathrm{b}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}})$

$$\mathcal{F}_{\mathcal{S}_k} \simeq \mathsf{E}_{\mathcal{S}_k|\mathbb{C}_z}^{-\operatorname{Re}\frac{c}{2}z^2} \oplus \mathsf{E}_{\mathcal{S}_k|\mathbb{C}_z}^{-\operatorname{Re}\frac{d}{2}z^2}.$$

We remark that this is indeed the correct order of the direct summands: It is not difficult to see geometrically that $|c| < |d|\cos(\arg d - \arg c)$, and this is equivalent to $c <_{\theta_0} d$.

In the sequences and diagrams in this section, we will write direct sums associated to such a decomposition vertically, i.e. the summands one above the other. The gluing morphism for the transition from a sector $\mathcal{S}_k$ to $\mathcal{S}_{k+1}$ is given by the matrix σ_k. We will mimic the approach of Section 3.3. To simplify notation, we will suppress the second index in the exponentials, i.e. write E_Z^φ instead of $\mathsf{E}_{Z|\mathbb{C}}^\varphi$, for instance.

The upper and lower half-planes

One has a short exact sequence of enhanced sheaves

$$0 \longrightarrow \mathcal{F}_{\mathcal{H}_+} \longrightarrow \mathcal{F}_{\mathcal{S}_1} \oplus \mathcal{F}_{\mathcal{S}_2} \longrightarrow \mathcal{F}_{\mathcal{S}_{12}} \longrightarrow 0,$$

which is isomorphic to

$$0 \longrightarrow \mathcal{F}_{\mathcal{H}_+} \longrightarrow \begin{matrix} \mathsf{E}_{\mathcal{S}_1}^{-\operatorname{Re}\frac{c}{2}z^2} \\ \oplus \\ \mathsf{E}_{\mathcal{S}_1}^{-\operatorname{Re}\frac{d}{2}z^2} \end{matrix} \oplus \begin{matrix} \mathsf{E}_{\mathcal{S}_2}^{-\operatorname{Re}\frac{c}{2}z^2} \\ \oplus \\ \mathsf{E}_{\mathcal{S}_2}^{-\operatorname{Re}\frac{d}{2}z^2} \end{matrix} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{matrix} \mathsf{E}_{\mathcal{S}_{12}}^{-\operatorname{Re}\frac{c}{2}z^2} \\ \oplus \\ \mathsf{E}_{\mathcal{S}_{12}}^{-\operatorname{Re}\frac{d}{2}z^2} \end{matrix} \longrightarrow 0. \tag{3.39}$$

Applying the enhanced Fourier–Sato transform and using the results found in Section 3.3.3 and Section 3.4.2, one obtains a distinguished triangle in $\mathrm{D}^\mathrm{b}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})[-1] \longrightarrow \begin{matrix} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi_\mathrm{r}^+ \rhd \varphi_\mathrm{r}^-} \\ \oplus \\ \mathsf{E}_{Y_1}^{\psi_\mathrm{r}^+ \rhd \psi_\mathrm{r}^-} \end{matrix} \oplus \begin{matrix} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi_\mathrm{l}^+ \rhd \varphi_\mathrm{l}^-} \\ \oplus \\ \mathsf{E}_{Y_2}^{\psi_\mathrm{l}^+ \rhd \psi_\mathrm{l}^-} \end{matrix} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \zeta} \end{matrix} \xrightarrow{+1}.$$

Remark. Let us give a short explanation why the morphism given by σ_1 in this triangle is well-defined (which might not be obvious at first sight): Note that $\psi_\mathrm{r}^+(w) = \psi_\mathrm{r}^-(w)$ for $w_2 = \frac{d_2}{d_1}w_1$ and $w_2 = \frac{c_2}{c_1}w_1$, so we could replace Y_1 by $\{w \in \mathbb{C}_w \mid w_2 < \min(\frac{c_2}{c_1}w_1, \frac{d_2}{d_1}w_1)\}$, which is an open subset of $\widehat{\mathcal{H}}_-$. This explains how nontrivial morphisms $\mathsf{E}_{Y_1}^{\psi_\mathrm{r}^+ \rhd \psi_\mathrm{r}^-} \to \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0\rhd\eta}$ and $\mathsf{E}_{Y_1}^{\psi_\mathrm{r}^+ \rhd \psi_\mathrm{r}^-} \to \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0\rhd\zeta}$ make sense (cf. Lemma A.4). For the morphism $\mathsf{E}_{Y_2}^{\psi_\mathrm{l}^+ \rhd \psi_\mathrm{l}^-} \to \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0\rhd\zeta}$, we note that $\widehat{\mathcal{H}}_-$ is a closed subset of Y_2.

Since the second and third object of this distinguished triangle are concentrated in degree 0, the associated long exact sequence in $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$ reduces to

$$0 \longrightarrow \mathrm{H}^{-1}\big({}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})\big) \longrightarrow \begin{matrix} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi_{\mathrm{r}}^+ \rhd \varphi_{\mathrm{r}}^-} \\ \oplus \\ \mathsf{E}_{Y_1}^{\psi_{\mathrm{r}}^+ \rhd \psi_{\mathrm{r}}^-} \end{matrix} \oplus \begin{matrix} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{\varphi_{\mathrm{l}}^+ \rhd \varphi_{\mathrm{l}}^-} \\ \oplus \\ \mathsf{E}_{Y_2}^{\psi_{\mathrm{l}}^+ \rhd \psi_{\mathrm{l}}^-} \end{matrix} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{H}}_-}^{0 \rhd \zeta} \end{matrix} \longrightarrow \mathrm{H}^{0}\big({}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})\big) \longrightarrow 0, \tag{3.40}$$

and it follows that $\mathrm{H}^l\big({}^{\mathcal{L}}(\mathcal{F}_{\widehat{\mathcal{H}}_+})\big) \simeq 0$ for $l \notin \{-1,0\}$. The following lemma implies that $\mathrm{H}^0\big({}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})\big) \simeq 0$.

Lemma 3.26. *The morphism $\sigma_1 - \mathbb{1}$ from (3.40) is an epimorphism in* $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$.

Proof. First, note that $\zeta(w) > \eta(w)$ for any $w \in \mathbb{C}_w$: Consider their difference

$$\zeta(w) - \eta(w) = \frac{1}{2}(c_1 w_2 - c_2 w_1)^2 \, \frac{c_1^2 d_1 - c_2^2 d_1 + 2c_1c_2d_2 - c_1^3 - c_1c_2^2}{c_1|c|^2(c_1^2 d_1 - c_2^2 d_1 + 2c_1c_2d_2)}.$$

For the sums appearing in the fraction we can estimate

$$\begin{aligned} c_1^2 d_1 - c_2^2 d_1 + 2c_1c_2d_2 &> c_1^2 d_1 - c_2^2 d_1 + 2c_1c_2d_2 - c_1^3 - c_1c_2^2 \\ &> c_1^2 d_1 - c_2^2 d_1 + 2c_1c_2d_2 - c_1^2 d_1 - c_1c_2d_2 \\ &= c_2(c_1d_2 - c_2d_1) \geq 0. \end{aligned}$$

This follows from the conditions imposed on c and d (cf. ($\llcorner$)). In particular, in the second line, we have used $c_1 < d_1$ and $c_2 < d_2$. This shows that the enumerator and the denominator are both strictly positive and hence $\zeta > \eta$.

Moreover, one has $\zeta(w) \leq \mathrm{Re}\,\frac{1}{2d}w^2$, which can be seen from the equality

$$\mathrm{Re}\,\frac{1}{2d}w^2 - \zeta(w) = \frac{\big((c_1d_1 + c_2d_2)w_1 + (c_1d_2 - c_2d_1)w_2\big)^2}{2|d|^2(c_1^2 d_1 - c_2^2 d_1 + 2c_1c_2d_2)},$$

using again the estimate from above to see that the denominator is positive.

We check the surjectivity of the morphism $\sigma_1 - \mathbb{1}$ on stalks and distinguish several cases. Let $(\check{w}, \check{t}) \in \mathbb{C}_w \times \mathbb{R}$. We only need to consider the cases where $\check{w} \in \widehat{\mathcal{H}}_-$ and $0 \leq \check{t} < -\eta$ since otherwise the stalk of the target sheaf is zero and the morphism is clearly surjective. Let us treat the cases in a detailed manner here.

<u>Case 1:</u> $\check{w} \in \widehat{\mathcal{H}}_-$, $(c_1d_2 - c_2d_1)\check{w}_2 > -(c_1d_1 + c_2d_2)\check{w}_1$ and $\check{w}_1 > 0$.

If $0 \leq \check{t} < -\zeta(\check{w})$ (which also implies $\check{t} < -\eta(\check{w})$ by what we have shown above), the induced map on stalks is

$$\begin{matrix} \mathbf{k} \\ \oplus \\ \mathbf{k} \end{matrix} \oplus \begin{matrix} * \\ \oplus \\ * \end{matrix} \longrightarrow \begin{matrix} \mathbf{k} \\ \oplus \\ \mathbf{k}, \end{matrix}$$

$$\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \longmapsto \sigma_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where $* \in \{0, \mathbf{k}\}$ denotes a trivial or one-dimensional $\mathbf{k}$-vector space (and it does not have to be the same each time $*$ appears). Since σ_1 is invertible, this map is surjective and a preimage of $(v_1, v_2)^{\mathrm{T}}$ is given by $\big(\sigma_1^{-1}(v_1, v_2)^{\mathrm{T}}, (0,0)^{\mathrm{T}}\big)$.

If $-\zeta(\check{w}) \leq \check{t} < -\eta(\check{w})$, the situation is similar since the induced map is

$$\begin{array}{ccc} \mathbf{k} & & * \\ \oplus & \oplus & \oplus \\ 0 & & 0 \end{array} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{array}{c} \mathbf{k} \\ \oplus \\ 0, \end{array}$$

$$\left(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}\right) \longmapsto \sigma_1 \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

(Note that this is well-defined since σ_1 is upper-triangular.) This map is surjective, and preimages can be found as before since σ_1^{-1} is still upper-triangular.

Case 2: $\check{w} \in \widehat{\mathcal{H}}_-$, $(c_1d_2 - c_2d_1)\check{w}_2 \leq -(c_1d_1 + c_2d_2)\check{w}_1$ and $\check{w}_1 > 0$.
(It is shown at the end of Section B.3.1 that $0 \leq -\operatorname{Re} \frac{1}{2d}\check{w}^2$ under these assumptions.)

If $0 \leq \check{t} < -\operatorname{Re} \frac{1}{2d}\check{w}^2$ (and hence also $\check{t} < -\zeta(\check{w})$), the induced morphism on stalks is

$$\begin{array}{ccc} \mathbf{k} & & * \\ \oplus & \oplus & \oplus \\ \mathbf{k} & & \mathbf{k} \end{array} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{array}{c} \mathbf{k} \\ \oplus \\ \mathbf{k}, \end{array}$$

which is clearly surjective.

If $-\operatorname{Re} \frac{1}{2d}\check{w}^2 \leq \check{t} < -\zeta(\check{w})$, we get

$$\begin{array}{ccc} \mathbf{k} & & * \\ \oplus & \oplus & \oplus \\ 0 & & \mathbf{k} \end{array} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{array}{c} \mathbf{k} \\ \oplus \\ \mathbf{k}, \end{array}$$

and a preimage of $(v_1, v_2)^{\mathrm{T}}$ is given by $\big(\sigma_1^{-1}(v_1, 0)^{\mathrm{T}}, (0, -v_2)^{\mathrm{T}}\big)$.

If $-\zeta(\check{w}) \leq \check{t} < -\eta(\check{w})$ (and hence $-\operatorname{Re} \frac{1}{2d}\check{w}^2 \leq \check{t}$), one obtains the map

$$\begin{array}{ccc} \mathbf{k} & & * \\ \oplus & \oplus & \oplus \\ 0 & & 0 \end{array} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{array}{c} \mathbf{k} \\ \oplus \\ 0, \end{array}$$

whose surjectivity is obvious.

Case 3: $\check{w} \in \widehat{\mathcal{H}}_-$ and $\check{w}_1 \leq 0$.

If $0 \leq \check{t} < -\zeta(\check{w})$, the induced map on stalks is

$$\begin{array}{ccc} * & & \mathbf{k} \\ \oplus & \oplus & \oplus \\ * & & \mathbf{k} \end{array} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{array}{c} \mathbf{k} \\ \oplus \\ \mathbf{k}, \end{array}$$

and if $-\zeta(\check{w}) \leq \check{t} < -\eta(\check{w})$, it is given by

$$\begin{array}{ccc} * & & \mathbf{k} \\ \oplus & \oplus & \oplus \\ * & & 0 \end{array} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{array}{c} \mathbf{k} \\ \oplus \\ 0. \end{array}$$

Both of these maps are easily seen to be surjective. $\square$

In view of the sequence (3.40), the lemma implies that $\mathrm{H}^0\big({}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})\big) \simeq 0$, and hence ${}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})$ is concentrated in degree -1. More precisely, we have proved the following proposition.

Proposition 3.27. *There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$

$${}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_+})[-1] \simeq \ker \left(\sigma_1 - \mathbb{1} \colon \begin{array}{c} \mathsf{E}^{\varphi_{\mathrm{r}}^+ \rhd \varphi_{\mathrm{r}}^-}_{\widehat{\mathcal{H}}_-} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{r}}^+ \rhd \psi_{\mathrm{r}}^-}_{Y_1} \end{array} \oplus \begin{array}{c} \mathsf{E}^{\varphi_{\mathrm{l}}^+ \rhd \varphi_{\mathrm{l}}^-}_{\widehat{\mathcal{H}}_-} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{l}}^+ \rhd \psi_{\mathrm{l}}^-}_{Y_2} \end{array} \longrightarrow \begin{array}{c} \mathsf{E}^{0 \rhd \eta}_{\widehat{\mathcal{H}}_-} \\ \oplus \\ \mathsf{E}^{0 \rhd \zeta}_{\widehat{\mathcal{H}}_-} \end{array} \right).$$

In a completely analogous way, starting from the sequence

$$0 \longrightarrow \mathcal{F}_{\mathcal{H}_-} \longrightarrow \mathcal{F}_{\mathcal{S}_4} \oplus \mathcal{F}_{\mathcal{S}_3} \longrightarrow \mathcal{F}_{\mathcal{S}_{34}} \longrightarrow 0,$$

one obtains the following result.

Proposition 3.28. *There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$

$${}^{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_-})[-1] \simeq \ker \left(\mathbb{1} - \sigma_3 \colon \begin{array}{c} \mathsf{E}^{\varphi_{\mathrm{r}}^+ \rhd \varphi_{\mathrm{r}}^-}_{\widehat{\mathcal{H}}_+} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{r}}^+ \rhd \psi_{\mathrm{r}}^-}_{Y_4} \end{array} \oplus \begin{array}{c} \mathsf{E}^{\varphi_{\mathrm{l}}^+ \rhd \varphi_{\mathrm{l}}^-}_{\widehat{\mathcal{H}}_+} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{l}}^+ \rhd \psi_{\mathrm{l}}^-}_{Y_3} \end{array} \longrightarrow \begin{array}{c} \mathsf{E}^{0 \rhd \eta}_{\widehat{\mathcal{H}}_+} \\ \oplus \\ \mathsf{E}^{0 \rhd \zeta}_{\widehat{\mathcal{H}}_+} \end{array} \right).$$

The upper and lower half-planes $\mathcal{H}_+$ and $\mathcal{H}_-$ intersect in the horizontal real line $L := \mathcal{H}_+ \cap \mathcal{H}_- = \mathcal{S}_{41} \cup \mathcal{S}_{23}$. One has the short exact sequence of enhanced sheaves

$$0 \longrightarrow \mathcal{F}_L \longrightarrow \mathcal{F}_{\mathcal{S}_{41}} \oplus \mathcal{F}_{\mathcal{S}_{23}} \longrightarrow \mathcal{F}_{\{0\}} \longrightarrow 0.$$

We use the trivializing isomorphism from $\mathcal{S}_1$ for $\mathcal{F}_{\mathcal{S}_{41}}$ and $\mathcal{F}_{\{0\}}$, and the one from $\mathcal{S}_3$ for $\mathcal{S}_{23}$. The sequence then reads as

$$0 \longrightarrow \mathcal{F}_L \longrightarrow \begin{array}{c} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{41}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{41}} \end{array} \oplus \begin{array}{c} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{23}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{23}} \end{array} \xrightarrow{\mathbb{1} - \sigma_4\sigma_3} \begin{array}{c} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\{0\}} \end{array} \longrightarrow 0, \tag{3.41}$$

and its enhanced Fourier–Sato transform (up to a shift) is the distinguished triangle

$$
{}^{\mathcal{L}}(\mathcal{F}_L)[-1] \longrightarrow \begin{matrix} \mathsf{E}^{\varphi_{\mathrm{r}}^+}_{\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{r}}^+}_{\mathbb{C}_w|\mathbb{C}_w} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\varphi_{\mathrm{l}}^+}_{\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{l}}^+}_{\mathbb{C}_w} \end{matrix} \xrightarrow{\mathbb{1}-\sigma_4\sigma_3} \begin{matrix} \mathsf{E}^{0}_{\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\mathbb{C}_w} \end{matrix} \xrightarrow{+1} .
$$

As in the previous cases, we show that the morphism $\mathbb{1} - \sigma_4\sigma_3$ is an epimorphism and obtain the following result.

Proposition 3.29. *There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w\times\mathbb{R}})$

$$
{}^{\mathcal{L}}(\mathcal{F}_L)[-1] \simeq \ker\left(\mathbb{1} - \sigma_4\sigma_3 \colon \begin{matrix} \mathsf{E}^{\varphi_{\mathrm{r}}^+}_{\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{r}}^+}_{\mathbb{C}_w} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\varphi_{\mathrm{l}}^+}_{\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\psi_{\mathrm{l}}^+}_{\mathbb{C}_w} \end{matrix} \longrightarrow \begin{matrix} \mathsf{E}^{0}_{\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\mathbb{C}_w} \end{matrix} \right).
$$

For the Fourier–Sato transform of the objects supported on the origin, we use Proposition 3.13. This proposition is also valid for the parameter d since at the origin the functions $\operatorname{Re} \frac{c}{2}z^2$ and $\operatorname{Re} \frac{d}{2}z^2$ agree.

The complex plane

We will now put the above results together to determine ${}^{\mathcal{L}}\mathcal{F}$. Consider the short exact sequence in $\mathrm{Mod}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}})$

$$
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{\mathcal{H}_+} \oplus \mathcal{F}_{\mathcal{H}_-} \longrightarrow \mathcal{F}_L \longrightarrow 0. \tag{3.42}
$$

Note that the (canonical) morphism $\mathcal{F}_{\mathcal{H}_+} \to \mathcal{F}_L$ fits into a commutative diagram in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_z\times\mathbb{R}})$ combining (3.39) and (3.41)

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\mathcal{H}_+} \arrow[r] \arrow[d] & \begin{matrix} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_1} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_1} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_2|\mathbb{C}_z} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_2} \end{matrix} \arrow[r, "\sigma_1-\mathbb{1}"] \arrow[d, "\mathbb{1}|\sigma_2"] & \begin{matrix} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{12}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{12}} \end{matrix} \arrow[r] \arrow[d, "\sigma_1^{-1}"] & 0 \\
0 \arrow[r] & \mathcal{F}_L \arrow[r] & \begin{matrix} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{41}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{41}} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\mathcal{S}_{23}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\mathcal{S}_{23}} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \begin{matrix} \mathsf{E}^{-\operatorname{Re}\frac{c}{2}z^2}_{\{0\}} \\ \oplus \\ \mathsf{E}^{-\operatorname{Re}\frac{d}{2}z^2}_{\{0\}} \end{matrix} \arrow[r] & 0.
\end{tikzcd}
$$

The vertical arrows are not all canonical (i.e. given by identities) since, for instance, the trivializing isomorphism on $\mathcal{S}_{23}$ is induced by that on $\mathcal{S}_3$. The vertical arrow in the middle denotes the direct sum of the morphism given by $\mathbb{1}$ between the summands on the left and the morphism given by σ_2 between the summands on the right. Similarly, the morphism $\mathcal{F}_{\mathcal{H}_-} \to \mathcal{F}_L$ can be found in such a diagram whose middle arrow is given by $\sigma_4|\mathbb{1}$.

We will show that the Fourier–Sato transform of $\mathcal{F}$ decomposes as a direct sum of exponentials

$$\mathsf{E}_{\widehat{\mathcal{S}}_k}^{\mathrm{Re}\,\frac{1}{2c}w^2} \oplus \mathsf{E}_{\widehat{\mathcal{S}}_k}^{\mathrm{Re}\,\frac{1}{2d}w^2}$$

on the sectors defined as follows:

$$\begin{aligned}
\widehat{\mathcal{S}}_1 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[-\pi + \arg c, -\frac{\pi}{2}\right] \text{ if } w \neq 0 \right\}, \\
\widehat{\mathcal{S}}_2 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[-\frac{\pi}{2}, \arg c\right] \text{ if } w \neq 0 \right\}, \\
\widehat{\mathcal{S}}_3 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[\arg c, \frac{\pi}{2}\right] \text{ if } w \neq 0 \right\}, \\
\widehat{\mathcal{S}}_4 &:= \left\{ w \in \mathbb{C}_w \;\middle|\; \arg w \in \left[\frac{\pi}{2}, \pi + \arg c\right] \text{ if } w \neq 0 \right\}.
\end{aligned}$$

The exponents in this decomposition can be written as $-\mathrm{Re}\,\frac{\widehat{c}}{2}w^2$ and $-\mathrm{Re}\,\frac{\widehat{d}}{2}w^2$ if we set $\widehat{c} := -\frac{1}{c}$ and $\widehat{d} := -\frac{1}{d}$. This indicates that ${}^{\mathcal{L}}\mathcal{F}$ is again of pure Gaussian type with parameter set $\widehat{C} = -1/C = \{\widehat{c}, \widehat{d}\}$.

Let us discuss why these sectors are suitable, i.e. why each of them contains exactly one Stokes direction: The Stokes lines for $\widehat{C}$ are the solutions for $\arg w$ of

$$\mathrm{Re}\left(\left(-\frac{1}{c} + \frac{1}{d}\right)w^2\right) = 0.$$

This is equivalent to

$$\mathrm{Re}\left(\frac{d-c}{cd}w^2\right) = 0$$

and hence to (recalling that $\arg(d-c) = \omega$)

$$\cos(\omega - \arg c - \arg d + 2\arg w) = 0.$$

The Stokes directions are therefore

$$\widehat{\mathrm{st}}_k = \frac{3\pi}{4} + \frac{1}{2}\arg c + \frac{1}{2}\arg d - \frac{\omega}{2} + k\frac{\pi}{2}$$

for $k \in \mathbb{Z}/4\mathbb{Z}$. In order to show that $\widehat{\mathrm{st}}_k$ is contained in $\widehat{\mathcal{S}}_k$ (and in no other sector), it suffices to show that this is the case for $k = 3$. (The rest will then be clear since $\widehat{\mathcal{S}}_3$ is acute-angled and the Stokes directions differ by multiples of $\frac{\pi}{2}$.) This follows from $\arg c \leq \omega < \frac{\pi}{2}$ and $\arg c \leq \arg d < \frac{\pi}{2}$ by the computations

$$\widehat{\mathrm{st}}_3 = \frac{\pi}{4} + \frac{1}{2}\arg c + \frac{1}{2}\arg d - \frac{\omega}{2} > \frac{1}{2}\arg c + \frac{1}{2}\arg d \geq \arg c$$

and

$$\widehat{\mathrm{st}}_3 = \frac{\pi}{4} + \frac{1}{2}\arg c + \frac{1}{2}\arg d - \frac{\omega}{2} \leq \frac{\pi}{4} + \frac{1}{2}\arg d < \frac{\pi}{2}.$$

The direction $\widehat{\theta}_0 = \pi + \frac{1}{2}\arg c$ is generic because there are no Stokes directions with arguments in $[\pi, \pi + \arg c]$ by the computations just performed. Indeed, we obtain that $\widehat{\mathrm{st}}_4 \in (\frac{\pi}{2} + \arg c, \pi)$ and $\widehat{\mathrm{st}}_1 \in (\pi + \arg c, \frac{3\pi}{2})$, so $\widehat{\theta}_0$ lies between these two Stokes directions. Therefore, once we have trivializations of ${}^{\mathcal{L}}\mathcal{F}$ on the sectors $\widehat{\mathcal{S}}_k$, one automatically gets trivializations (with the same gluing matrices) on the sectors $\widehat{S}_k = \{w \in \mathbb{C} \mid \arg w \in [\widehat{\theta}_0 + (k-1)\frac{\pi}{2}, \widehat{\theta}_0 + k\frac{\pi}{2}] \text{ if } w \neq 0\}$ (see Lemma 3.9). One can then conclude that ${}^{\mathcal{L}}\mathcal{F}$ is of pure Gaussian type $\widehat{C}$.

Furthermore, we have $\widehat{c} <_{\widehat{\theta}_0} \widehat{d}$ (this is equivalent to $|d| > |c|\cos(\arg d - \arg c)$, which is again clear geometrically), the order on $\widehat{C}$ with respect to $\widehat{\theta}_0$ is the one induced by the order on C with respect to θ_0, meaning that the order of the direct summands does not change when we write the exponents in terms of $\widehat{C}$.

The sectors $\widehat{\mathcal{S}}_k$ are defined analogously to those in Section 3.3. The difference and difficulty is that the sets Y_k are not unions of these sectors as this was the case for $\widehat{\mathcal{H}}_+$ and $\widehat{\mathcal{H}}_-$ (cf. Fig. 3.8).

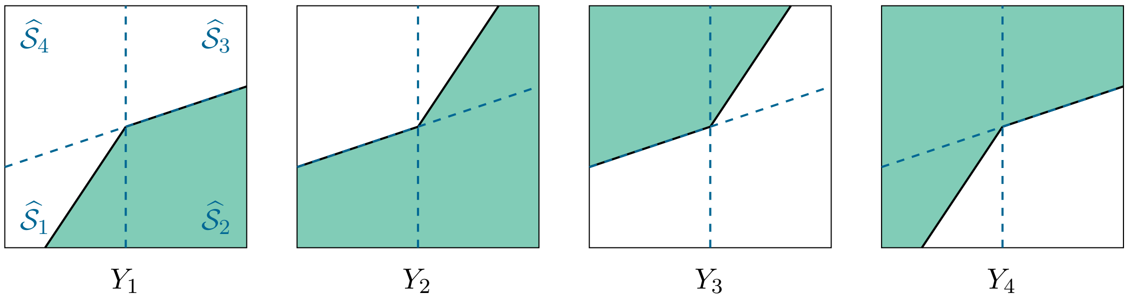


Figure 3.8.: The sets Y_k and their relative positions with respect to the sectors $\widehat{\mathcal{S}}_k$:
The green regions Y_k are bounded by two half-lines with slopes $\frac{c_2}{c_1}$ and $\frac{d_2}{d_1}$ (the latter being the steeper one due to condition (∟)).
The blue lines indicate the borders of the sectors $\widehat{\mathcal{S}}_k$.

Proposition 3.30. *The enhanced Fourier–Sato transform of $\mathcal{F}$ is an enhanced sheaf of pure Gaussian type $\widehat{C} = \{-\frac{1}{c}, -\frac{1}{d}\}$. In other words, the object ${}^{\mathcal{L}}\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$ is concentrated in degree 0 and there are isomorphisms*

$$({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_k} \simeq \mathsf{E}_{\widehat{\mathcal{S}}_k}^{\operatorname{Re}\frac{1}{2c}w^2} \oplus \mathsf{E}_{\widehat{\mathcal{S}}_k}^{\operatorname{Re}\frac{1}{2d}w^2}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{C}_w \times \mathbb{R}})$ for any $k \in \mathbb{Z}/4\mathbb{Z}$.

Proof. Applying the enhanced Fourier–Sato transform and the functor $(\bullet)_{\widehat{\mathcal{S}}_k}$ to the sequence (3.42) yields a distinguished triangle, whose associated long exact sequence reduces to

$$
\begin{aligned}
0 \longrightarrow \mathrm{H}^{-1}({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_k} \longrightarrow \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_k} \oplus \ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_k} \\
\longrightarrow \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_k} \longrightarrow \mathrm{H}^{0}({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}_k}} \longrightarrow 0
\end{aligned}
\tag{3.43}
$$

in view of Propositions 3.27, 3.28 and 3.29. In particular, it follows that $\mathrm{H}^l({}^{\mathcal{L}}\mathcal{F}) \simeq 0$ for $l \notin \{-1, 0\}$.

As in the case of aligned parameters, one can describe the kernels appearing in this sequence more explicitly on sectors. Let us treat the case $k = 1$ in detail. Since kernels commute with exact functors, $\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1}$ is the kernel of the morphism

$$
\begin{matrix}
\mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2}_{Y_1 \cap \widehat{\mathcal{S}}_1}
\end{matrix}
\oplus
\begin{matrix}
\mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_1}
\end{matrix}
\xrightarrow{\sigma_1 - \mathbb{1}}
\begin{matrix}
\mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_1}
\end{matrix}.
$$

We will now prove that the sequence

$$
0 \longrightarrow
\begin{matrix}
\mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2}_{Y_1 \cap \widehat{\mathcal{S}}_1}
\end{matrix}
\xrightarrow{(\mathbb{1}, \sigma_1)}
\begin{matrix}
\mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2}_{Y_1 \cap \widehat{\mathcal{S}}_1}
\end{matrix}
\oplus
\begin{matrix}
\mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_1}
\end{matrix}
\xrightarrow{\sigma_1 - \mathbb{1}}
\begin{matrix}
\mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_1} \\
\oplus \\
\mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_1}
\end{matrix}
\longrightarrow 0
\tag{3.44}
$$

is exact, and we check this on stalks: Let $(\check{w}, \check{t}) \in \mathbb{C}_w \times \mathbb{R}$. Assume moreover that $\check{w} \in \widehat{\mathcal{S}}_1$. (Otherwise, the stalk at $(\check{w}, \check{t})$ of all the objects in the sequence is zero and exactness is clear.) If $\check{t} < 0$, the induced sequence on stalks is

$$
0 \longrightarrow V \xrightarrow{(\mathrm{id}, 0)} V \oplus 0 \longrightarrow 0 \longrightarrow 0
$$

for some (at most two-dimensional) $\mathbf{k}$-vector space V, and this sequence is obviously exact.

If $\check{t} \geq 0$, the stalk at $(\check{w}, \check{t})$ of the sequence (3.46) is

$$
0 \longrightarrow
\begin{matrix} \bigstar_c \\ \oplus \\ \bigstar_d \end{matrix}
\xrightarrow{(\mathbb{1}, \sigma_1)}
\begin{matrix} \bigstar_c \\ \oplus \\ \bigstar_d \end{matrix}
\oplus
\begin{matrix} \blacktriangle_c \\ \oplus \\ \blacktriangle_d \end{matrix}
\xrightarrow{\sigma_1 - \mathbb{1}}
\begin{matrix} \blacktriangle_c \\ \oplus \\ \blacktriangle_d \end{matrix}
\longrightarrow 0,
$$

where $\bigstar_c = \mathbf{k}$ if $\check{t} < -\operatorname{Re} \frac{1}{2c} \check{w}^2$, and $\bigstar_c = 0$ otherwise. Similarly, $\bigstar_d = \mathbf{k}$ if $\check{t} < -\operatorname{Re} \frac{1}{2d} \check{w}^2$ and $\check{w} \in Y_1$, and $\bigstar_d = 0$ otherwise. Likewise, $\blacktriangle_c = \mathbf{k}$ if $\check{t} < -\eta(\check{w})$, and $\blacktriangle_c = 0$ otherwise, $\blacktriangle_d = \mathbf{k}$ if $\check{t} < -\zeta(\check{w})$, and $\blacktriangle_d = 0$ otherwise. Note that, recalling $\eta \leq \operatorname{Re} \frac{1}{2c} w^2$, $\zeta \leq \operatorname{Re} \frac{1}{2d} w^2$ and $\eta < \zeta$, we have $\bigstar_d = 0$ whenever $\blacktriangle_d = 0$, and similarly $\bigstar_c = 0$ whenever $\blacktriangle_c = 0$, and

$\blacktriangle_d = 0$ whenever $\blacktriangle_c = 0$, so the maps given by σ_1 are indeed well-defined. It is then clear that the sequence is exact because the kernel of $\sigma_1 - \mathbb{1}$ obviously consists precisely of all the elements of the form $(v, \sigma_1(v))$.

Thus we have proved the exactness of (3.46), and we obtain an isomorphism

$$\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \simeq \begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{Y_1 \cap \widehat{\mathcal{S}}_1} \end{array}.$$

Similarly, $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1}$ is the kernel of the morphism

$$\begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_{41}} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{Y_4 \cap \widehat{\mathcal{S}}_1} \end{array} \oplus \begin{array}{c} \mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_{41}} \\ \oplus \\ \mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_{41}} \end{array} \xrightarrow{\mathbb{1} - \sigma_3} \begin{array}{c} \mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_{41}} \\ \oplus \\ \mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_{41}} \end{array},$$

and one notes that almost all the objects are zero since $\frac{w_1^2}{2c_1} = \mathrm{Re}\,\frac{1}{2c} w^2$ and $0 = \eta(w) = \zeta(w)$ for $c_1 w_2 - c_2 w_1 = 0$ (and in particular for $w \in \widehat{\mathcal{S}}_{41}$). The morphism $\mathbb{1} - \sigma_3$ is therefore just the zero morphism in this case, and it fits into the short exact sequence

$$0 \longrightarrow \begin{array}{c} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{Y_4 \cap \widehat{\mathcal{S}}_1} \end{array} \xrightarrow{-\sigma_4^{-1}} \begin{array}{c} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{Y_4 \cap \widehat{\mathcal{S}}_1} \end{array} \longrightarrow \begin{array}{c} 0 \\ \oplus \\ 0 \end{array} \longrightarrow 0, \tag{3.45}$$

proving that we have an isomorphism

$$\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1} \simeq \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{Y_4 \cap \widehat{\mathcal{S}}_1}.$$

As shown in the sequence, we will keep the upper summand in our notation although it is zero, since we want to stick to using 2×2 matrices for our morphisms. Note that in the sequence (3.45) one could also choose the identity matrix (or, more generally, any lower triangular matrix with a nonzero entry in the lower right corner) as the morphism. However, we chose $-\sigma_4^{-1}$ in order to make the diagram below less complicated.

In the same manner, the short exact sequence

$$0 \longrightarrow \begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_1} \end{array} \xrightarrow{(\mathbb{1}, \sigma_2\sigma_1)} \begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_1} \end{array} \oplus \begin{array}{c} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \end{array} \xrightarrow{\mathbb{1} - \sigma_4\sigma_3} \begin{array}{c} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \end{array} \longrightarrow 0$$

yields an isomorphism

$$\ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1} \simeq \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2c_1}} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\frac{w_1^2}{2d_1}} \end{array}.$$

We now put the three above short exact sequences together to form the columns of a commutative diagram whose upper row will contain (under the isomorphisms constructed) the morphism

$$\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_1} \oplus \ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_1} \longrightarrow \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_1}$$

(i.e. the central part of (3.43)) given by the Fourier–Laplace transform of the morphisms in the sequence (3.42). The diagram we obtain is shown on p. 101.

The columns of this diagram are exact by construction and it is easily checked on stalks that the first row is also exact, in particular that the third object is really the cokernel of the morphism induced by the second row. If we compare the first row to the sequence (3.43), which was the object of our study, it follows that $({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1}$ is concentrated in degree 0, and we obtain the desired isomorphism

$$({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_1} \simeq \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\operatorname{Re}\frac{1}{2c}w^2} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_1}^{\operatorname{Re}\frac{1}{2d}w^2} \end{array}.$$

$\square$

Diagram for $\widehat{\mathcal{S}}_1$:

$$
\begin{tikzcd}[column sep=large]
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} & & 0 \\ \oplus & \oplus & \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_1\cap\widehat{\mathcal{S}}_1} & & \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_4\cap\widehat{\mathcal{S}}_1} \end{matrix} \arrow[r, "\mathbb{1}+\mathbb{1}"] \arrow[d, "(\mathbb{1}{,}\sigma_1)"', "-\sigma_4^{-1}"] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_1} \end{matrix} \arrow[r, "\mathbb{1}"] \arrow[d, "(\mathbb{1}{,}\sigma_2\sigma_1)"] & \begin{matrix} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_1} \end{matrix} \arrow[r] & 0 \\
 & \begin{pmatrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1} & & \mathsf{E}^{0\rhd\eta}_{\widehat{\mathcal{S}}_1} \\ \oplus & \oplus & \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_1\cap\widehat{\mathcal{S}}_1} & & \mathsf{E}^{0\rhd\zeta}_{\widehat{\mathcal{S}}_1} \end{pmatrix} \oplus \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_4\cap\widehat{\mathcal{S}}_1} \end{matrix} \arrow[r, "(\mathbb{1}|\sigma_2)-(\sigma_4{,}0)"] \arrow[d, "\sigma_1-\mathbb{1}"', "0"] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_1} & & \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \\ \oplus & \oplus & \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_1} & & \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \end{matrix} \arrow[d, "\mathbb{1}-\sigma_4\sigma_3"] & & \\
 & \begin{matrix} \mathsf{E}^{0\rhd\eta}_{\widehat{\mathcal{S}}_1} & & 0 \\ \oplus & \oplus & \oplus \\ \mathsf{E}^{0\rhd\zeta}_{\widehat{\mathcal{S}}_1} & & 0 \end{matrix} \arrow[r, "\sigma_1^{-1}+0"] \arrow[d] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_1} \end{matrix} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Remark. Let us point out two facts about the diagram on p. 101 which may not be obvious and even confusing at first sight.

- One may wonder why the lower square is commutative: Starting from the first summand, this is not difficult to see since $(\mathbb{1}-\sigma_4\sigma_3)\circ(\mathbb{1}|\sigma_2) = \mathbb{1}-\sigma_4\sigma_3\sigma_2 = \mathbb{1}-\sigma_1^{-1} = (\sigma_1-\mathbb{1})\circ\sigma_1^{-1}$. However, for the second summand, following the path to the right and then down, we obtain $-(\mathbb{1}-\sigma_4\sigma_3)\circ(\sigma_4,0) = -\sigma_4$. On the other hand, following the path first down and then to the right, we obtain $\sigma_1^{-1}\circ 0 = 0$, the zero morphism. The solution to this apparent contradiction is as follows: Actually, one can show that on $Y_4\cap\widehat{\mathcal{S}}_1$ (i.e. for $\check{w}_1\leq 0$ and $\frac{d_2}{d_1}\check{w}_1\leq\check{w}_2\leq\frac{c_2}{c_1}\check{w}_1$) we have $-\operatorname{Re}\frac{1}{2d}w^2<0$ (this follows easily from Lemma B.3), so any morphism
$$\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{Y_4\cap\widehat{\mathcal{S}}_1}\to\mathsf{E}^0_{\widehat{\mathcal{S}}_1}$$
is zero, and hence also the one represented by $-\sigma_4$.

- It is possible to write the first row of the diagram in an even nicer way: Note that $(Y_1\cap\widehat{\mathcal{S}}_1)\cup(Y_4\cap\widehat{\mathcal{S}}_1)=\widehat{\mathcal{S}}_1$ and one has $\frac{w_1^2}{2d_1}=\operatorname{Re}\frac{1}{2d}w^2$ on the half-line $\Lambda := (Y_1\cap\widehat{\mathcal{S}}_1)\cap(Y_4\cap\widehat{\mathcal{S}}_1)=\{w\in\mathbb{C}_w\mid d_2w_1-d_1w_2=0, w_1\leq 0\}$. Hence, we get isomorphisms
$$\begin{aligned}\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{Y_1\cap\widehat{\mathcal{S}}_1}\oplus\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{Y_4\cap\widehat{\mathcal{S}}_1}&\simeq\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{(Y_1\cap\widehat{\mathcal{S}}_1)\smallsetminus\Lambda}\oplus\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{(Y_4\cap\widehat{\mathcal{S}}_1)\smallsetminus\Lambda}\\&\xrightarrow{\sim}\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_1\smallsetminus\Lambda}\\&\simeq\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_1},\end{aligned}$$
where the morphism in the second line is the given by the sum of the two canonical morphisms (which exist since $(Y_1\cap\widehat{\mathcal{S}}_1)\smallsetminus\Lambda$ and $(Y_4\cap\widehat{\mathcal{S}}_1)\smallsetminus\Lambda$ are open subsets of $\widehat{\mathcal{S}}_1\smallsetminus\Lambda$). The object in the upper left corner of the diagram could therefore be rewritten as
$$\begin{array}{c}\mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright\operatorname{Re}\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_1}\\\oplus\\\mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright\operatorname{Re}\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_1},\end{array}$$
making it even more similar to the diagram we obtained in the case of aligned parameters (see p. 75). The horizontal morphism $\mathbb{1}+\mathbb{1}$ would then simply be replaced by the morphism given by $\mathbb{1}$. The vertical morphism, however, is a bit more complicated to describe, and we will not go into detail here.

Remark. Similarly, one proves Proposition 3.30 on the other sectors. We will give some details for the case $k = 2$:

One finds the kernel $\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_2}$ as the first object in the short exact sequence

$$0 \longrightarrow \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2} \end{matrix} \xrightarrow{(\sigma_1^{-1}, \mathbb{1})} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0 \triangleright \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0 \triangleright \psi_{\mathrm{r}}^-} \end{matrix} \oplus \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2d_1} \triangleright \psi_{\mathrm{l}}^-} \end{matrix} \xrightarrow{\sigma_1 - \mathbb{1}} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0 \triangleright \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0 \triangleright \zeta} \end{matrix} \longrightarrow 0. \tag{3.46}$$

Unfortunately, we cannot replace the functions ψ_{r}^- and ψ_{l}^- by more explicit expressions here since $\widehat{\mathcal{S}}_2$ contains points where $(c_1 d_2 - c_2 d_1) w_2 \leq -(c_1 d_1 + c_2 d_2) w_1$ as well as points with $(c_1 d_2 - c_2 d_1) w_2 > -(c_1 d_1 + c_2 d_2) w_1$. Describing the situation colloquially, this means that, at the half-line where $(c_1 d_2 - c_2 d_1) w_2 = -(c_1 d_1 + c_2 d_2) w_1$, the exponents ζ and $\operatorname{Re} \frac{1}{2d} w^2$ seem to "change places". The kernel, however, is not "sensitive" to this change and is as we would have expected. The exactness of this sequence can be checked as usual: Consider the stalks at any point $(\check{w}, \check{t})$ and distinguish the cases where $\check{t} < 0$, $\check{t} < -\zeta(\check{w})$, $-\zeta(\check{w}) \leq \check{t} < -\eta(\check{w})$ and $-\eta(\check{w}) \leq \check{t}$, and distinguish additionally the cases $(c_1 d_2 - c_2 d_1) \check{w}_2 \leq -(c_1 d_1 + c_2 d_2) \check{w}_1$ and $(c_1 d_2 - c_2 d_1) \check{w}_2 > -(c_1 d_1 + c_2 d_2) \check{w}_1$.

The kernel $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_2}$ is the kernel of the morphism

$$\begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_{23}}^{0 \triangleright \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_{23}}^{0 \triangleright \zeta} \end{matrix} \oplus \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_{23}}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \\ \oplus \\ \mathsf{E}_{\{0\}}^{0 \triangleright 0} \end{matrix} \xrightarrow{\mathbb{1} - \sigma_3} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_{23}}^{0 \triangleright \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_{23}}^{0 \triangleright \zeta} \end{matrix}.$$

Note that $\frac{w_1^2}{2c_1} = \operatorname{Re} \frac{1}{2c} w^2$ and $\eta = \zeta = 0$ on $\widehat{\mathcal{S}}_{23}$, hence all the objects are zero. Therefore, also $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_2} \simeq 0$.

Finally, $\ker(\mathbb{1} - \sigma_4 \sigma_3)_{\widehat{\mathcal{S}}_2}$ is given by the first object in the short exact sequence

$$0 \longrightarrow \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1}} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2d_1}} \end{matrix} \xrightarrow{(\sigma_4 \sigma_3, \mathbb{1})} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0} \end{matrix} \oplus \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2c_1}} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{\frac{w_1^2}{2d_1}} \end{matrix} \xrightarrow{\mathbb{1} - \sigma_4 \sigma_3} \begin{matrix} \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_2}^{0} \end{matrix} \longrightarrow 0.$$

Again, one builds a big diagram out of all these sequences, yielding an isomorphism as desired. The procedure is analogous for the sectors $\widehat{\mathcal{S}}_3$ and $\widehat{\mathcal{S}}_4$. The corresponding diagrams are shown on pp. 104–106.

Diagram for $\widehat{\mathcal{S}}_2$:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_2} \arrow[r, "\sigma_2"] \arrow[d, "(\sigma_1^{-1}{,}\mathbb{1})"] & \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2} \arrow[r, "\sigma_2^{-1}"] \arrow[d, "(\sigma_4\sigma_3{,}\mathbb{1})"] & \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_2|\mathbb{C}_w} \arrow[r] & 0 \\
 & \left(\mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{0 \triangleright \psi_{\mathrm{r}}^-}_{\widehat{\mathcal{S}}_2}\right) \oplus \left(\mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \psi_{\mathrm{l}}^-}_{\widehat{\mathcal{S}}_2}\right) \arrow[r, "\mathbb{1}|\sigma_2"] \arrow[d, "\sigma_1 - \mathbb{1}"] & \left(\mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2}\right) \oplus \left(\mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2}\right) \arrow[d, "\mathbb{1} - \sigma_4\sigma_3"] & & \\
 & \mathsf{E}^{0 \triangleright \eta}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{0 \triangleright \zeta}_{\widehat{\mathcal{S}}_2} \arrow[r, "\sigma_1^{-1}"] \arrow[d] & \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Diagram for $\widehat{\mathcal{S}}_3$:

$$
\begin{tikzcd}[column sep=large]
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \begin{smallmatrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_2\cap\widehat{\mathcal{S}}_3} \end{smallmatrix} \oplus \begin{smallmatrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_3\cap\widehat{\mathcal{S}}_3} \end{smallmatrix} \arrow[r, "\mathbb{1}+\mathbb{1}"] \arrow[d, "\sigma_2^{-1}"', "{(-\sigma_3,-\mathbb{1})}"] & \begin{smallmatrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \arrow[r, "\mathbb{1}"] \arrow[d, "{(\sigma_4\sigma_3,\mathbb{1})}"] & \begin{smallmatrix} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \arrow[r] & 0 \\
 & \begin{smallmatrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_2\cap\widehat{\mathcal{S}}_3} \end{smallmatrix} \oplus \left(\begin{smallmatrix} \mathsf{E}^{0\rhd\eta}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0\rhd\zeta}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \oplus \begin{smallmatrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_3\cap\widehat{\mathcal{S}}_3} \end{smallmatrix} \right) \arrow[r, "(0{,}\sigma_2)-(\sigma_4|\mathbb{1})"] \arrow[d, "0"', "\mathbb{1}-\sigma_3"] & \begin{smallmatrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \oplus \begin{smallmatrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \arrow[d, "\mathbb{1}-\sigma_4\sigma_3"] & & \\
 & \begin{smallmatrix} 0 \\ \oplus \\ 0 \end{smallmatrix} \oplus \begin{smallmatrix} \mathsf{E}^{0\rhd\eta}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0\rhd\zeta}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \arrow[r, "0-\sigma_4"] \arrow[d] & \begin{smallmatrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \end{smallmatrix} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Diagram for $\widehat{\mathcal{S}}_4$:

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2} \end{array} \arrow[r, "\sigma_4"] \arrow[d, "(-\mathbb{1}{,}-\sigma_3^{-1})"] & \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1}} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2d_1}} \end{array} \arrow[r, "\sigma_4^{-1}"] \arrow[d, "(\mathbb{1}{,}\sigma_2\sigma_1)"] & \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\operatorname{Re} \frac{1}{2c} w^2} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\operatorname{Re} \frac{1}{2d} w^2} \end{array} \arrow[r] & 0 \\
& \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2d_1} \triangleright \psi_{\mathrm{r}}^-} \end{array} \oplus \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0 \triangleright \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0 \triangleright \psi_{\mathrm{l}}^-} \end{array} \arrow[r, "-(\sigma_4|\mathbb{1})"] \arrow[d, "\mathbb{1}-\sigma_3"] & \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2c_1}} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{\frac{w_1^2}{2d_1}} \end{array} \oplus \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0} \end{array} \arrow[d, "\mathbb{1}-\sigma_4\sigma_3"] & & \\
& \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0 \triangleright \eta} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0 \triangleright \zeta} \end{array} \arrow[r, "-\sigma_4"] \arrow[d] & \begin{array}{c} \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0} \\ \oplus \\ \mathsf{E}_{\widehat{\mathcal{S}}_4}^{0} \end{array} \arrow[d] & & \\
& 0 & 0 & &
\end{tikzcd}
$$

Stokes multipliers of the Fourier–Laplace transform

Finally, we want to compute the gluing isomorphisms of ${}^{\mathcal{L}}\mathcal{F}$. As seen in Proposition 3.30, one has isomorphisms

$$\widehat{\alpha}_k \colon ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_k} \xrightarrow{\simeq} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_k} \oplus \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_k}.$$

We fix such an isomorphism for any $k \in \mathbb{Z}/4\mathbb{Z}$. Better stated, we take the concrete isomorphisms constructed in the proof of Proposition 3.30. Then, on $\widehat{\mathcal{S}}_{k,k+1}$ one has the two isomorphisms

$$\widehat{\alpha}_k, \widehat{\alpha}_{k+1} \colon ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_{k,k+1}} \xrightarrow{\simeq} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{k,k+1}} \oplus \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_{k,k+1}},$$

induced by the above isomorphisms on $\widehat{\mathcal{S}}_k$ and $\widehat{\mathcal{S}}_{k+1}$, respectively (and denoted by the same symbols, by abuse of notation). The composition $\widehat{\alpha}_{k+1} \circ \widehat{\alpha}_k^{-1}$ is hence an automorphism of $\mathsf{E}^{\mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_{k,k+1}} \oplus \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_{k,k+1}}$, and the matrix representing this morphism (cf. Proposition 2.9) is denoted by $\widehat{\sigma}_k$. We obtain the following result.

Proposition 3.31. *Gluing matrices for ${}^{\mathcal{L}}\mathcal{F}$ are given by $\widehat{\sigma}_k = \sigma_k$ for any $k \in \mathbb{Z}/4\mathbb{Z}$.*

Proof. In order to find a transition matrix $\widehat{\sigma}_k$, we need to compare the diagrams on $\widehat{\mathcal{S}}_k$ and $\widehat{\mathcal{S}}_{k+1}$. More precisely, one starts by making more explicit the construction of the isomorphisms from Proposition 3.30. Let us treat the case $k = 2$.

The isomorphism

$$\ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_2} \simeq \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \oplus \mathsf{E}^{\frac{w_1^2}{2d_1} \rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_2}$$

was obtained by considering (3.46) together with Proposition 3.27, i.e. it is the first vertical morphism in the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, "\simeq"] & \begin{array}{c} \mathsf{E}^{0 \rhd \eta}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0 \rhd \psi_{\mathrm{r}}^-}_{\widehat{\mathcal{S}}_2} \end{array} \oplus \begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \rhd \psi_{\mathrm{l}}^-}_{\widehat{\mathcal{S}}_2} \end{array} \arrow[r, "\sigma_1 - \mathbb{1}"] \arrow[d, equal] & \begin{array}{c} \mathsf{E}^{0 \rhd \eta}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0 \rhd \zeta}_{\widehat{\mathcal{S}}_2} \end{array} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \rhd \mathrm{Re}\,\frac{1}{2d}w^2}_{\widehat{\mathcal{S}}_2} \end{array} \arrow[r, "(\sigma_1^{-1}{,}\mathbb{1})"] & \begin{array}{c} \mathsf{E}^{0 \rhd \eta}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0 \rhd \psi_{\mathrm{r}}^-}_{\widehat{\mathcal{S}}_2} \end{array} \oplus \begin{array}{c} \mathsf{E}^{\frac{w_1^2}{2c_1} \rhd \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \rhd \psi_{\mathrm{l}}^-}_{\widehat{\mathcal{S}}_2} \end{array} \arrow[r, "\sigma_1 - \mathbb{1}"] & \begin{array}{c} \mathsf{E}^{0 \rhd \eta}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0 \rhd \zeta}_{\widehat{\mathcal{S}}_2} \end{array} \arrow[r] & 0.
\end{tikzcd}$$

The kernel from Proposition 3.28 was shown to be zero on $\widehat{\mathcal{S}}_2$, so there is a unique isomorphism $\ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_2} \xrightarrow{\simeq} 0$.

Similarly to the first one, the kernel from Proposition 3.29 was made more explicit on $\widehat{\mathcal{S}}_2$ by the coloured isomorphism in the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, "\simeq"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] \arrow[d, equal] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r, "{(\sigma_4\sigma_3,\mathbb{1})}"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r] & 0.
\end{tikzcd}
\tag{3.47}
$$

In the diagram on $\widehat{\mathcal{S}}_2$ (see p. 104), the first part of the upper row is isomorphic to

$$
0 \longrightarrow \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_2} \oplus \ker(\mathbb{1}-\sigma_3)_{\widehat{\mathcal{S}}_2} \longrightarrow \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2}
$$

by the coloured isomorphisms from above. Therefore, we were able to compare this upper row to the exact sequence (3.43) and obtained the desired isomorphism $\widehat{\alpha}_2$ as the blue arrow in the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_2} \oplus \ker(\mathbb{1}-\sigma_3)_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, "\simeq"] & \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_2} \arrow[r] \arrow[d, "\simeq", "\widehat{\alpha}_1"'] & 0 \\
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_2} \end{matrix} \oplus \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \arrow[r, "\sigma_2+0"] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r, "\sigma_2^{-1}"] & \begin{matrix} \mathsf{E}^{\operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_2} \\ \oplus \\ \mathsf{E}^{\operatorname{Re} \frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r] & 0.
\end{tikzcd}
\tag{3.48}
$$

We can construct a similar diagram for $\widehat{\mathcal{S}}_3$:

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1-\mathbb{1})_{\widehat{\mathcal{S}}_3} \oplus \ker(\mathbb{1}-\sigma_3)_{\widehat{\mathcal{S}}_3} \arrow[r] \arrow[d, dashed, "\simeq"] & \ker(\mathbb{1}-\sigma_4\sigma_3)_{\widehat{\mathcal{S}}_3} \arrow[r] \arrow[d, dashed, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_3} \arrow[r] \arrow[d, dashed, "\simeq", "\widehat{\alpha}_3"'] & 0 \\
0 \arrow[r] & \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2}_{Y_2 \cap \widehat{\mathcal{S}}_3} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1} \triangleright \operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \operatorname{Re} \frac{1}{2d} w^2}_{Y_3 \cap \widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "\mathbb{1}+\mathbb{1}"] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "\mathbb{1}"] & \begin{matrix} \mathsf{E}^{\operatorname{Re} \frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\operatorname{Re} \frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] & 0.
\end{tikzcd}
\tag{3.49}
$$

This is obtained by putting together the coloured isomorphisms from the diagrams

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_3} \arrow[r] \arrow[d, dashed, "\simeq"] & \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_2\cap\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] \arrow[d, equal] & \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_2\cap\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "\sigma_2"] & \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_2\cap\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] & \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \arrow[r] & 0,
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_3} \arrow[r] \arrow[d, dashed, "\simeq"] & \begin{matrix} \mathsf{E}^{0\triangleright\eta}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0\triangleright\zeta}_{\widehat{\mathcal{S}}_3} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_3\cap\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_3"] \arrow[d, equal] & \begin{matrix} \mathsf{E}^{0\triangleright\eta}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0\triangleright\zeta}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_3\cap\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "(-\sigma_3{,}-\mathbb{1})"] & \begin{matrix} \mathsf{E}^{0\triangleright\eta}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0\triangleright\zeta}_{\widehat{\mathcal{S}}_3} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}\triangleright \mathrm{Re}\,\frac{1}{2c}w^2}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}\triangleright \mathrm{Re}\,\frac{1}{2d}w^2}_{Y_3\cap\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_3"] & \begin{matrix} \mathsf{E}^{0\triangleright\eta}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0\triangleright\zeta}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] & 0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_3} \arrow[r] \arrow[d, dashed, "\simeq"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_2} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] \arrow[d, equal] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "(\sigma_4\sigma_3{,}\mathbb{1})"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r, "\mathbb{1}-\sigma_4\sigma_3"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_3} \end{matrix} \arrow[r] & 0.
\end{tikzcd}
\tag{3.50}
$$

Finally, we restrict diagrams (3.48) and (3.49) to the half-line $\widehat{\mathcal{S}}_{23}$ (by applying the functor $(\bullet)_{\widehat{\mathcal{S}}_{23}}$) and identify their first lines, which yields

$$
\begin{tikzcd}
0 \arrow[r] & \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_{23}} \end{matrix} \oplus \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \arrow[r, "\sigma_2 + 0"] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}} \end{matrix} \arrow[r, "\sigma_2^{-1}"] \arrow[dd, bend left, orange] & \begin{matrix} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_{23}} \\ \oplus \\ \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_{23}} \end{matrix} \arrow[r] \arrow[dd, bend left, "\widehat{\sigma}_2"] & 0 \\
0 \arrow[r] & \ker(\sigma_1 - \mathbb{1})_{\widehat{\mathcal{S}}_{23}} \oplus \ker(\mathbb{1} - \sigma_3)_{\widehat{\mathcal{S}}_{23}} \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_{23}} \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & ({}^{\mathcal{L}}\mathcal{F})_{\widehat{\mathcal{S}}_{23}} \arrow[r] \arrow[u, "\simeq", "\widehat{\alpha}_2"'] \arrow[d, dashed, "\simeq", "\widehat{\alpha}_3"'] & 0 \\
0 \arrow[r] & \begin{matrix} 0 \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1} \triangleright \mathrm{Re}\,\frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_{23}} \end{matrix} \oplus \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \arrow[r, "\mathbb{1} + 0"] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}} \end{matrix} \arrow[r, "\mathbb{1}"] & \begin{matrix} \mathsf{E}^{\mathrm{Re}\,\frac{1}{2c} w^2}_{\widehat{\mathcal{S}}_{23}} \\ \oplus \\ \mathsf{E}^{\mathrm{Re}\,\frac{1}{2d} w^2}_{\widehat{\mathcal{S}}_{23}} \end{matrix} \arrow[r] & 0.
\end{tikzcd}
\tag{3.51}
$$

Our aim is to find the matrix $\widehat{\sigma}_2$ representing the purple arrow, which is nothing but the transition morphism $\widehat{\alpha}_3 \circ \widehat{\alpha}_2^{-1}$. To describe it, we have to determine the orange arrow. The latter can be found as follows: Combine diagrams (3.47) and (3.50) by applying the functor $(\bullet)_{\widehat{\mathcal{S}}_{23}}$, and note that the first rows of both diagrams will then be the same (since $w_1 = 0$ on $\widehat{\mathcal{S}}_{23}$). One therefore obtains the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r, "{(\sigma_4\sigma_3, \mathbb{1})}"] \arrow[dd, bend left, orange] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r, "\mathbb{1} - \sigma_4\sigma_3"] \arrow[d, equal] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\mathbb{1} - \sigma_4\sigma_3)_{\widehat{\mathcal{S}}_{23}} \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r, "\mathbb{1} - \sigma_4\sigma_3"] \arrow[d, equal] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r, "{(\sigma_4\sigma_3, \mathbb{1})}"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \oplus \begin{matrix} \mathsf{E}^{\frac{w_1^2}{2c_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{\frac{w_1^2}{2d_1}}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r, "\mathbb{1} - \sigma_4\sigma_3"] & \begin{matrix} \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \\ \oplus \\ \mathsf{E}^{0}_{\widehat{\mathcal{S}}_{23}|\mathbb{C}_w} \end{matrix} \arrow[r] & 0,
\end{tikzcd}
$$

where one finds the orange arrow again. Due to commutativity, the latter must be given by the identity matrix $\mathbb{1}$. Altogether, considering diagram (3.51) again, the purple arrow is represented by the matrix σ_2, as claimed.

The other cases work completely analogously, and this completes the proof. $\square$

We have thus finished the proof of Theorem 3.22.

To end with, let us reflect on the meaning of the results and on the applicability to further cases:

In this section, we have shown that the considerations of Section 3.3 can – with a little effort, but without serious difficulties – be adapted to more general situations.

Although we have only proved Theorem 3.22 for two parameters, a corresponding statement is proved analogously for more than two parameters: For example, in the case of three parameters c, d and e, we would need to require that (c, d) and (d, e) satisfy condition ($\llcorner$) (and hence automatically (c, e) satisfies this condition).

Condition ($\llcorner$) was chosen in such a way that results from Section 3.3 could be reused in the computations, and such that the geometry of the hyperbolae and sectors involved is still reasonable to describe. However, we expect other cases to be treatable with these methods. Under different assumptions on the parameters, just as in our cases, one needs to choose suitable sectors in the domain and target, compute the enhanced Fourier–Sato transform of exponentials on these sectors and describe the global picture by means of short exact sequences.

Appendices

Appendix A.

Sheaves on locally closed subsets

Let X always be a nonempty and connected good topological space. Denote by $\mathbf{k}$ a field and by $\mathbf{k}_X$ the constant sheaf with stalk $\mathbf{k}$ on X. In this appendix, we will clarify some properties of sheaves supported on locally compact subsets and prove some statements used in the thesis.

Let us start with a lemma. An important fact following directly from the behaviour of the six operations on sheaves in the presence of a Cartesian square is the following.

Lemma A.1. *(i) Let $j\colon Z \hookrightarrow X$ be a homeomorphism onto a locally closed subspace. Then we have a natural isomorphism $j^{-1}j_! \simeq \mathrm{id}_{\mathrm{D^b}(\mathbf{k}_Z)}$.*

(ii) Let $h\colon X \to Y$ be a homeomorphism with inverse $\tilde{h}\colon Y \to X$. Then we have a natural isomorphism of functors $\tilde{h}^{-1} \simeq h_!$.

Proof. The first statement follows from the Cartesian diagram

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, equal] & Z \arrow[d, hook, "j"] \\ Z \arrow[r, hook, "j"] & X \end{tikzcd}$$

and [21, Proposition 2.6.7]. (Note that j^{-1} and $j_!$ are exact.)

For the second statement, we note that $\tilde{h}^{-1}\tilde{h}_! \simeq \mathrm{id}_{\mathrm{D^b}(\mathbf{k}_Y)}$ by (i). Moreover, we have $\tilde{h} \circ h = \mathrm{id}_X$, and hence

$$\tilde{h}^{-1} \simeq \tilde{h}^{-1}\big(\tilde{h} \circ h\big)_! \simeq \big(\tilde{h}^{-1}\tilde{h}_!\big)h_! \simeq h_!.$$

□

Recall that, for a sheaf $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_X)$ and a locally closed subset $Z \subseteq X$ with inclusion $j\colon Z \hookrightarrow X$, we set $\mathcal{F}_Z := j_!j^{-1}\mathcal{F}$. Recall moreover that we write $\mathbf{k}_Z$ instead of $(\mathbf{k}_X)_Z$. The stalk of $\mathbf{k}_Z$ is $\mathbf{k}$ at any point of Z and 0 at any point of $X \smallsetminus Z$. For locally closed sets $Z, Z_1, Z_2 \subseteq X$ and a continuous map $f\colon Y \to X$, one has $f^{-1}\mathbf{k}_Z \simeq \mathbf{k}_{f^{-1}(Z)}$ and $\mathbf{k}_{Z_1} \otimes \mathbf{k}_{Z_2} \simeq \mathbf{k}_{Z_1 \cap Z_2}$ (see e.g. [21, Chapter II]).

A.1. Morphisms

It is well-known that $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_X, \mathbf{k}_X) \simeq \mathbf{k}$ (this follows easily from [21, Proposition 2.3.10], for example), where we write for short $\mathrm{Hom}_{\mathbf{k}_X}$ instead of $\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_X)}$. Concretely, every endomorphism of the constant sheaf $\mathbf{k}_X$ is given by multiplication with an element $a \in \mathbf{k}$ on global sections $\mathbf{k}_X(X) \simeq \mathbf{k}$.

For sheaves of the form $\mathbf{k}_Z$, where Z is open or closed, one can describe quite explicitly the morphisms between them, and the (rather strict) conditions under which nontrivial morphisms exist.

Lemma A.2. *Let $A, B \subseteq X$ be closed and let $U, V \subseteq X$ be open subsets. Assume moreover that A, B, U, V are nonempty and connected. Then*

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_A, \mathbf{k}_B) \simeq \begin{cases} \mathbf{k} & \text{if } B \subseteq A, \\ 0 & \text{otherwise} \end{cases},$$

and

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_U, \mathbf{k}_V) \simeq \begin{cases} \mathbf{k} & \text{if } U \subseteq V, \\ 0 & \text{otherwise} \end{cases}.$$

Proof. The first assertion is proved as follows: Denote by $b\colon B \hookrightarrow X$ the embedding. Note that b is proper and hence $b_! \simeq b_*$. Let $B \subseteq A$. Then

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_A, \mathbf{k}_B) \simeq \mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_A, b_!\mathbf{k}_B) \simeq \mathrm{Hom}_{\mathbf{k}_B}(b^{-1}\mathbf{k}_A, \mathbf{k}_B) \simeq \mathrm{Hom}_{\mathbf{k}_B}(\mathbf{k}_B, \mathbf{k}_B) \simeq \mathbf{k}.$$

On the other hand, if $B \nsubseteq A$, we have

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_A, \mathbf{k}_B) \simeq \Gamma(X; \Gamma_A(\mathbf{k}_B)) \simeq \Gamma(B; \Gamma_{A\cap B}(\mathbf{k}_B)) \simeq \ker\big(\mathbf{k}_B(B) \to \mathbf{k}_B(B \setminus A)\big)$$

by [21, (2.3.16), (2.3.20), Proposition 2.3.9 (i) and (2.3.14)]. By our assumption, $B \setminus A$ is not empty. We know that $\mathbf{k}_B(B) \simeq \mathbf{k}$ (which can be interpreted as the vector space of constant functions $B \to \mathbf{k}$). Similarly, $\mathbf{k}_B(B \setminus A)$ is the vector space of locally constant functions $B \setminus A \to \mathbf{k}$, and the morphism $\mathbf{k}_B(B) \to \mathbf{k}_B(B \setminus A)$ is just restriction of functions. Therefore, the morphism is not zero and hence its kernel vanishes (since its domain is one-dimensional), as claimed.

Next, we prove the second assertion: Denote by $u\colon U \hookrightarrow X$ the embedding. Note that $u^! \simeq u^{-1}$ since u is an open embedding. Let $U \subseteq V$. Then

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_U, \mathbf{k}_V) \simeq \mathrm{Hom}_{\mathbf{k}_X}(u_!\mathbf{k}_U, \mathbf{k}_V) \simeq \mathrm{Hom}_{\mathbf{k}_U}(\mathbf{k}_U, u^{-1}\mathbf{k}_V) \simeq \mathrm{Hom}_{\mathbf{k}_U}(\mathbf{k}_U, \mathbf{k}_U) \simeq \mathbf{k}.$$

In contrast, if $U \nsubseteq V$, we get

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_U, \mathbf{k}_V) \simeq \Gamma(X; \Gamma_U(\mathbf{k}_V)) \simeq \mathbf{k}_V(U)$$

by [21, (2.3.16) and Proposition 2.3.9 (i)] and the definition of $\Gamma_U(\mathbf{k}_V)$. Since the functor

$\Gamma(U; \bullet)$ taking sections on U is left exact, we have a short exact sequence

$$0 \longrightarrow \mathbf{k}_V(U) \longrightarrow \mathbf{k}_X(U) \longrightarrow \mathbf{k}_{X\smallsetminus V}(U).$$

We claim that the last morphism in this sequence is injective: We know that $\mathbf{k}_X(U) \simeq \mathbf{k}$ (constant functions on U) and that $\mathbf{k}_{X\smallsetminus V}(U) \simeq \mathbf{k}_{X\smallsetminus V}(U \smallsetminus V)$ (see [21, Proposition 2.3.6 (iv)]) is the vector space of locally constant functions on $U \smallsetminus V$. Hence the argument works as above. Consequently, $\mathbf{k}_V(U) \simeq 0$. □

Under the assumptions of this lemma, there is a canonical (nontrivial) morphism $\mathbf{k}_A \to \mathbf{k}_B$ for $B \subseteq A$ closed and $\mathbf{k}_U \to \mathbf{k}_V$ for $U \subseteq V$ open, defined by the identity on $\mathbf{k}_B$ and $\mathbf{k}_U$, respectively. This morphism is identified with the element $1 \in \mathbf{k}$, which gives a canonical choice for the isomorphisms $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_A, \mathbf{k}_B) \simeq \mathbf{k}$ and $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_U, \mathbf{k}_V) \simeq \mathbf{k}$ in these cases. The morphism given by $a \in \mathbf{k}$ then induces multiplication by a on stalks (at points of B or U, or the zero morphism if one of the stalks is trivial).

If $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{Z_1}, \mathbf{k}_{Z_2}) \simeq \mathbf{k}$, $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{Z_2}, \mathbf{k}_{Z_3}) \simeq \mathbf{k}$ and $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{Z_1}, \mathbf{k}_{Z_3}) \simeq \mathbf{k}$, then – under the identifications from above – the composition is given by multiplication. Note, however, that it can happen that $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{Z_1}, \mathbf{k}_{Z_2}) \simeq \mathbf{k}$ and $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{Z_2}, \mathbf{k}_{Z_3}) \simeq \mathbf{k}$, but $\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{Z_1}, \mathbf{k}_{Z_3}) \simeq 0$, in which case the composition is the zero morphism, of course. (This occurs, for instance, if $Z_1 \subseteq Z_2$ is open and $Z_3 \subseteq Z_2$ is closed, but $Z_1 \cap Z_3 = \varnothing$.)

Example A.3. It is possible to describe the morphisms from Lemma A.2 more intuitively and explicitly: Let $X = \mathbb{C}$ be the complex plane and $B \subset X$ a closed ball. Then $\mathbf{k}_B$ can be described as follows: It suffices to define the sections of the sheaf on open balls W (since these form a basis of the topology). We have

$$\mathbf{k}_B(W) = \begin{cases} \mathbf{k} & \text{if } B \cap W \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and restriction morphisms are identities $\mathbf{k} \to \mathbf{k}$ or zero morphisms $\mathbf{k} \to 0$ and $0 \to 0$.

Now let B_1, B_2 be two closed balls with $B_1 \subsetneq B_2$. The canonical morphism $\mathbf{k}_{B_2} \to \mathbf{k}_{B_1}$ is defined on sections by the identity $\mathbf{k} \to \mathbf{k}$ if $\mathbf{k}_{B_2}(W) = \mathbf{k}_{B_1}(W) = \mathbf{k}$ and zero morphisms otherwise. (Again, it suffices to define a morphism on open balls W.)

The fact that there is no nontrivial morphism $f \colon \mathbf{k}_{B_1} \to \mathbf{k}_{B_2}$ can be seen in the following way: Assume $f(W)$ is nontrivial for some open ball W, i.e. $f(W) \colon \mathbf{k} = \mathbf{k}_{B_1}(W) \to \mathbf{k}_{B_2}(W) = \mathbf{k}$ is multiplication by some element $a \in \mathbf{k}$. Then consider a smaller ball $W' \subset W$ with $W' \cap B_1 = \varnothing$ but $W' \cap B_2 \neq \varnothing$. Since a morphism of sheaves must be compatible with restrictions, we get a commutative diagram

$$\begin{array}{ccc} \mathbf{k} = \mathbf{k}_{B_1}(W) & \xrightarrow{a\,\cdot} & \mathbf{k}_{B_2}(W) = \mathbf{k} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{id}} \\ 0 = \mathbf{k}_{B_1}(W') & \xrightarrow{0} & \mathbf{k}_{B_2}(W') = \mathbf{k}, \end{array}$$

and this is clearly only commutative if $a = 0$.

Similar considerations can be made with sheaves of the form $\mathbf{k}_U$ for an open ball U, which is defined on open balls W by

$$\mathbf{k}_U(W) = \begin{cases} \mathbf{k} & \text{if } W \subseteq U. \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence of Lemma A.2, we have the following statement about morphisms of exponential enhanced sheaves (as introduced in Definition 1.3).

Lemma A.4. *Let $Z_1, Z_2 \subseteq X$ be locally closed and connected. Let $U \subseteq X$ be open with $Z_1, Z_2 \subseteq U$ and let $\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^- \colon U \to \mathbb{R}$ be continuous functions with $\varphi_1^+(x) > \varphi_1^-(x)$ for any $x \in Z_1$ and $\varphi_2^+(x) > \varphi_2^-(x)$ for any $x \in Z_2$. Then*

$$\mathrm{Hom}_{\mathbf{k}_{X\times\mathbb{R}}}\big(\mathsf{E}_{Z_1|X}^{\varphi_1^+ \rhd \varphi_1^-}, \mathsf{E}_{Z_2|X}^{\varphi_2^+ \rhd \varphi_2^-}\big) \simeq \mathbf{k}$$

if one of the following two conditions is satisfied:

(i) Z_2 is a closed subset of Z_1, and we have $\varphi_1^+ \geq \varphi_2^+$, $\varphi_1^- \geq \varphi_2^-$ and $\varphi_1^- < \varphi_2^+$ on Z_2.

(ii) Z_1 is an open subset of Z_2, and we have $\varphi_1^+ \geq \varphi_2^+$, $\varphi_1^- \geq \varphi_2^-$ and $\varphi_1^- < \varphi_2^+$ on Z_1.

Proof. We only prove (i). Since all the sheaves have support contained in $Z_1 \times \mathbb{R}$ and extension by zero is fully faithful, we can assume $X = Z_1$ without loss of generality. Consider the inclusion

$$j \colon (Z_2 \times \mathbb{R}) \cap \{-\varphi_2^+ \leq t\} \hookrightarrow Z_2 \times \mathbb{R}.$$

We write for short $Y := (Z_2 \times \mathbb{R}) \cap \{-\varphi_2^+ \leq t\}$. One then has

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{k}_{X\times\mathbb{R}}}\big(\mathsf{E}_{X|X}^{\varphi_1^+ \rhd \varphi_1^-}, \mathsf{E}_{Z_2|X}^{\varphi_2^+ \rhd \varphi_2^-}\big) &\simeq \mathrm{Hom}_{\mathbf{k}_{X\times\mathbb{R}}}\big(\mathbf{k}_{\{-\varphi_1^+ \leq t < -\varphi_1^-\}}, j_!\mathbf{k}_{(Z_2\times\mathbb{R})\cap\{-\varphi_2^+ \leq t < -\varphi_2^-\}}\big) \\
&\simeq \mathrm{Hom}_{\mathbf{k}_Y}\big(j^{-1}\mathbf{k}_{\{-\varphi_1^+ \leq t < -\varphi_1^-\}}, \mathbf{k}_{(Z_2\times\mathbb{R})\cap\{-\varphi_2^+ \leq t < -\varphi_2^-\}}\big) \\
&\simeq \mathrm{Hom}_{\mathbf{k}_Y}\big(\mathbf{k}_{(Z_2\times\mathbb{R})\cap\{-\varphi_2^+ \leq t < -\varphi_1^-\}}, \mathbf{k}_{(Z_2\times\mathbb{R})\cap\{-\varphi_2^+ \leq t < -\varphi_2^-\}}\big) \\
&\simeq \mathbf{k}.
\end{aligned}$$

The second isomorphism follows by adjunction since j is a closed embedding and hence $j_! \simeq j_*$. By our assumptions, the sets $(Z_2 \times \mathbb{R}) \cap \{-\varphi_2^+ \leq t < -\varphi_1^-\}$ and $(Z_2 \times \mathbb{R}) \cap \{-\varphi_2^+ \leq t < -\varphi_2^-\}$ are nonempty and connected, and the first is an open subset of the second. We then obtain the last isomorphism by Lemma A.2. □

A.2. Convolution

Recall the convolution product on $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times\mathbb{R}})$: One sets

$$\mathcal{F} \overset{*}{\otimes} \mathcal{G} := \mathrm{R}\mu_!\big(q_1^{-1}\mathcal{F} \otimes q_2^{-1}\mathcal{G}\big)$$

with the maps $\mu, q_1, q_2 \colon X \times \mathbb{R}^2 \to X \times \mathbb{R}$ given by $\mu(x, t_1, t_2) = (x, t_1 + t_2)$, $q_1(x, t_1, t_2) = (x, t_1)$ and $q_2(x, t_1, t_2) = (x, t_2)$. Recall moreover that we write $\{t \geq f\} := \{(x,t) \in X \times \mathbb{R} \mid x \in U, t \geq f(x)\}$ for $f \colon U \to \mathbb{R}$ a continuous function.

We will need the following result. In the context of ind-sheaves on bordered spaces, a similar statement can be found in [6, Lemma 4.2.3]. We will give a proof for enhanced sheaves here.

Lemma A.5. *Let $U_1, U_2 \subseteq X$ be open and let $f \colon U_1 \to \mathbb{R}$, $g \colon U_2 \to \mathbb{R}$ be continuous functions. Denote by $f + g \colon U_1 \cap U_2 \to \mathbb{R}, x \mapsto f(x) + g(x)$ the sum of the restrictions. Then there are isomorphisms in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X \times \mathbb{R}})$*

(i) $\mathbf{k}_{\{t \geq f\}} \overset{*}{\otimes} \mathbf{k}_{\{t \geq g\}} \simeq \mathbf{k}_{\{t \geq f+g\}}$.

(ii) $\mathbf{k}_{\{t \leq f\}} \overset{*}{\otimes} \mathbf{k}_{\{t \geq g\}} \simeq 0$.

Proof. First, we show that $\mathbf{k}_{\{t \geq f\}} \simeq \mathbf{k}_{\{t \geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t = f\}}$: We have

$$\mathbf{k}_{\{t \geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t = f\}} \simeq \mathrm{R}\mu_! \mathbf{k}_{\{t_1 \geq 0, t_2 = f\}}.$$

Consider the commutative diagram

$$\begin{tikzcd} \{t_1 \geq 0, t_2 = f\} \arrow[r, hook, "i"] \arrow[d, "s"] & X \times \mathbb{R}^2 \arrow[d, "\mu"] \\ \{t \geq f\} \arrow[r, hook, "j"] & X \times \mathbb{R}, \end{tikzcd}$$

which shows that $\mathrm{R}\mu_! i_! = j_! \mathrm{R}s_!$. (The extension by zero functors $i_!$ and $j_!$ are exact and we usually do not write them.) Note that s is a homeomorphism and therefore

$$\mathbf{k}_{\{t \geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t = f\}} \simeq j_! \mathrm{R}s_! \mathbf{k}_{\{t_1 \geq 0, t_2 = f\}} \simeq \mathbf{k}_{\{t \geq f\}}$$

as claimed.

Next, we note that $\mathbf{k}_{\{t = f\}} \overset{*}{\otimes} \mathbf{k}_{\{t = g\}} \simeq \mathbf{k}_{\{t = f+g\}}$: This is proved similarly to the above, considering the commutative diagram

$$\begin{tikzcd} \{t_1 = f, t_2 = g\} \arrow[r, hook, "i"] \arrow[d, "s"] & X \times \mathbb{R}^2 \arrow[d, "\mu"] \\ \{t = f + g\} \arrow[r, hook, "j"] & X \times \mathbb{R}, \end{tikzcd}$$

where again s is a homeomorphism.

Finally, we prove that $\mathbf{k}_{\{t \geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t \geq 0\}} \simeq \mathbf{k}_{\{t \geq 0\}}$: From the commutative square

$$\begin{tikzcd} \{t_1 \geq 0, t_2 \geq 0\} \arrow[r, hook, "i"] \arrow[d, "s"] & X \times \mathbb{R}^2 \arrow[d, "\mu"] \\ \{t \geq 0\} \arrow[r, hook, "j"] & X \times \mathbb{R} \end{tikzcd}$$

we see that
$$\mathbf{k}_{\{t\geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t\geq 0\}} \simeq \mathrm{R}\mu_!\mathbf{k}_{\{t_1\geq 0, t_2\geq 0\}} \simeq j_!\mathrm{R}s_!\mathbf{k}_{\{t_1\geq 0,t_2\geq 0\}}.$$
Now $\mathrm{R}s_!\mathbf{k}_{\{t_1\geq 0,t_2\geq 0\}} \simeq s_!\mathbf{k}_{\{t_1\geq 0,t_2\geq 0\}}$ since the stalks of the cohomologies are

$$\begin{aligned}
\big(\mathrm{R}^l s_!\mathbf{k}_{\{t_1\geq 0,t_2\geq 0\}}\big)_{(p,a)} &\simeq \mathrm{H}^l_{\mathrm{c}}\big(\{t_1\geq 0, t_2\geq 0\}; \mathbf{k}_{\{t_1\geq 0,t_2\geq 0\}}\big|_{s^{-1}(p,a)}\big)\\
&\simeq \mathrm{H}^l_{\mathrm{c}}\big(\{t_1\geq 0, t_2\geq 0\}; \mathbf{k}_{\{x=p,t_1\geq 0,t_2\geq 0,t_1+t_2=a\}}\big)\\
&\simeq \mathrm{H}^l_{\mathrm{c}}\big(\{x=p, t_1\geq 0, t_2\geq 0, t_1+t_2=a\}; \mathbf{k}\big)
\end{aligned}$$

for any $p\in X$, $a\geq 0$ and any $l\in\mathbb{Z}$ (cf. Lemma B.2) and the cohomology with compact support of a compact interval vanishes for $l\neq 0$. Since s is proper and the inverse image of the constant sheaf is constant, we get

$$\mathbf{k}_{\{t\geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t\geq 0\}} \simeq s_*s^{-1}\mathbf{k}_{\{t\geq 0\}} \simeq \mathbf{k}_{\{t\geq 0\}},$$

where the last isomorphism follows from the fact that by definition of direct and inverse images $(s_*s^{-1}\mathbf{k}_{\{t\geq 0\}})(U) = (s^{-1}\mathbf{k}_{\{t\geq 0\}})(s^{-1}(U)) = \mathbf{k}_{\{t\geq 0\}}(s(s^{-1}(U))) = \mathbf{k}_{\{t\geq 0\}}(U)$ (because $s(s^{-1}(U))$ is again U and in particular open).

This proves (i), using associativity and commutativity of convolution.

For the proof of (ii), it suffices to show that $\mathbf{k}_{\{t\geq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t\leq 0\}} \simeq 0$. We have

$$\mathbf{k}_{\{t\leq 0\}} \overset{*}{\otimes} \mathbf{k}_{\{t\geq 0\}} \simeq \mathrm{R}\mu_!\mathbf{k}_{\{t_1\leq 0,t_2\geq 0\}}$$

and the stalks of its cohomologies all vanish since

$$\big(\mathrm{R}^l\mu_!\mathbf{k}_{\{t_1\leq 0,t_2\geq 0\}}\big)_{(p,a)} \simeq \mathrm{H}^l_{\mathrm{c}}\big(\{x=p, t_1\leq 0, t_2\geq 0, t_1+t_2=a\}; \mathbf{k}\big)$$

for any $(p,a)\in X\times\mathbb{R}$ and any $l\in\mathbb{Z}$ (by a chain of isomorphisms as above), and we know that cohomology with compacts support of a closed half-line is zero (cf. Lemma B.1).

□

A.3. Gluing

A well-known construction is the gluing of sheaves defined on the elements of an open covering via isomorphisms on the intersections satisfying cocycle conditions (which ensure compatibility on triple intersections). In general, this does not work for closed coverings unless they are finite.

With the following lemma, we describe such a gluing in the special case of four closed sectors, a situation which often occurs in this work. The necessary gluing data consists only of four isomorphisms, and the cocycle conditions reduce to a single assumption on their composition.

Lemma A.6. *Given $\theta_0\in\mathbb{R}/4\mathbb{Z}$ and $R\in\mathbb{R}_{>0}$, let $S_k := \{z\in\mathbb{C} \mid |z|\leq R, \arg z\in [\theta_0+(k-1)\frac{\pi}{2}, \theta_0+k\frac{\pi}{2}] \text{ for } z\neq 0\}$, $k\in\mathbb{Z}/4\mathbb{Z}$, be four closed sectors (all of which contain* *the origin), and denote by* $S_{k,k+1} := S_k \cap S_{k+1}$ *their intersections. For any* $k \in \mathbb{Z}/4\mathbb{Z}$*, let* $\mathcal{F}_k \in \mathrm{Mod}(\mathbf{k}_{\mathbb{C}})$ *be a sheaf on* $\mathbb{C}$ *supported on*[7] S_k *and let*

$$\sigma_k\colon (\mathcal{F}_k)_{S_{k,k+1}} \xrightarrow{\sim} (\mathcal{F}_{k+1})_{S_{k,k+1}}$$

be an isomorphism of sheaves satisfying $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathrm{id}_{(\mathcal{F}_1)_{\{0\}}}$ *(where the left hand side denotes the composition of the induced isomorphisms* $\sigma_k\colon (\mathcal{F}_k)_{\{0\}} \xrightarrow{\sim} (\mathcal{F}_{k+1})_{\{0\}}$*). Then there exists a sheaf* $\mathcal{F}$ *on* $\mathbb{C}$*, unique up to unique isomorphism, supported on* $\bigcup_{k\in\mathbb{Z}/4\mathbb{Z}} S_k$*, together with isomorphisms* $\alpha_k\colon \mathcal{F}_{S_k} \xrightarrow{\sim} \mathcal{F}_k$ *such that (for the induced morphisms on* $S_{k,k+1}$*) one has* $\alpha_{k+1}\circ\alpha_k^{-1} = \sigma_k$*.*

Proof. We glue successively the sheaves $\mathcal{F}_k$. If $B \subseteq A \subseteq \mathbb{C}$ are closed subsets and $\mathcal{G}$ is a sheaf on $\mathbb{C}$, we will denote by "can" the morphism $\mathcal{G}_A \to \mathcal{G}_B$ obtained by tensoring the canonical morphism $\mathbf{k}_A \to \mathbf{k}_B$ (given by the element $1 \in \mathbf{k}$, cf. Lemma A.2) with $\mathcal{G}$.

(1) Glue $\mathcal{F}_{12}$ from $\mathcal{F}_1$ and $\mathcal{F}_2$:
First of all, define $\mathcal{F}_{12}$ to be a sheaf fitting into the short exact sequence

$$0 \longrightarrow \mathcal{F}_{12} \longrightarrow \mathcal{F}_1 \oplus \mathcal{F}_2 \xrightarrow{\sigma_1\circ\mathrm{can}-\mathrm{can}} (\mathcal{F}_2)_{S_{12}} \longrightarrow 0. \tag{A.1}$$

This sheaf is isomorphic to $\mathcal{F}_1$ and $\mathcal{F}_2$ on S_1 and S_2, respectively, since applying the functor $(\bullet)_{S_1}$ to (A.1) yields the first line of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F}_{12})_{S_1} \arrow[r] & \mathcal{F}_1 \oplus (\mathcal{F}_2)_{S_{12}} \arrow[r, "\sigma_1\circ\mathrm{can}-\mathrm{id}"] \arrow[d, "\mathrm{id}"', "\sigma_1^{-1}"] & (\mathcal{F}_2)_{S_{12}} \arrow[r] \arrow[d, "\sigma_1^{-1}"] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r, "(\mathrm{id},\mathrm{can})"] & \mathcal{F}_1 \oplus (\mathcal{F}_1)_{S_{12}} \arrow[r, "\mathrm{can}-\mathrm{id}"] & (\mathcal{F}_1)_{S_{12}} \arrow[r] & 0
\end{tikzcd} \tag{A.2}$$

and hence yields an isomorphism $(\mathcal{F}_{12})_{S_1} \simeq \mathcal{F}_1$ (exactness of the lower row is easily checked), and similarly for $(\mathcal{F}_{12})_{S_2}$.

(2) Glue $\mathcal{F}_{123}$ from $\mathcal{F}_{12}$ and $\mathcal{F}_3$:
For the next step of the gluing, we remark that the following diagram gives a gluing isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F}_{12})_{S_{23}} \arrow[r] \arrow[d, dashed, "\xi"', "\simeq"] & (\mathcal{F}_1)_{\{0\}} \oplus (\mathcal{F}_2)_{S_{23}} \arrow[r, "\sigma_1-\mathrm{can}"] \arrow[d, "\sigma_2\circ\sigma_1"', "\sigma_2"] & (\mathcal{F}_2)_{\{0\}} \arrow[r] \arrow[d, "\sigma_2"] & 0 \\
0 \arrow[r] & (\mathcal{F}_3)_{S_{23}} \arrow[r, "(\mathrm{can},\mathrm{id})"] & (\mathcal{F}_3)_{\{0\}} \oplus (\mathcal{F}_3)_{S_{23}} \arrow[r, "\mathrm{id}-\mathrm{can}"] & (\mathcal{F}_3)_{\{0\}} \arrow[r] & 0
\end{tikzcd} \tag{A.3}$$

(The first row of this diagram is obtained from (A.1) by applying $(\bullet)_{S_{23}}$.) The isomorphism ξ can also be obtained by applying the functor $(\bullet)_{S_{23}}$ to the isomorphism $(\mathcal{F}_{12})_{S_2} \simeq \mathcal{F}_2$ and composing with σ_2.

[7] By "$\mathcal{F}$ is supported on S" we mean $\operatorname{supp}\mathcal{F} \subseteq S$ (not necessarily equality).

We then define $\mathcal{F}_{123}$ as the sheaf fitting into the exact sequence

$$0 \longrightarrow \mathcal{F}_{123} \longrightarrow \mathcal{F}_{12} \oplus \mathcal{F}_3 \xrightarrow{\xi \circ \mathrm{can} - \mathrm{can}} (\mathcal{F}_3)_{S_{23}} \longrightarrow 0. \tag{A.4}$$

(3) Glue $\mathcal{F}$ from $\mathcal{F}_{123}$ and $\mathcal{F}_4$:
The last gluing isomorphism (on $S_4 \cap (S_1 \cup S_2 \cup S_3) = S_{41} \cup S_{34}$) is obtained from the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F}_{123})_{S_{41} \cup S_{34}} \arrow[r] \arrow[d, dashed, "\gamma"', "\simeq"] & (\mathcal{F}_1)_{S_{41}} \oplus (\mathcal{F}_3)_{S_{34}} \arrow[r, "\sigma_2 \circ \sigma_1 \circ \mathrm{can} - \mathrm{can}"] \arrow[d, "\sigma_4^{-1}"', "\sigma_3"] & (\mathcal{F}_3)_{\{0\}} \arrow[r] \arrow[d, "\sigma_3"] & 0 \\
0 \arrow[r] & (\mathcal{F}_4)_{S_{41} \cup S_{34}} \arrow[r, "(\mathrm{can}, \mathrm{id})"] & (\mathcal{F}_4)_{S_{41}} \oplus (\mathcal{F}_4)_{S_{34}} \arrow[r, "\mathrm{can} - \mathrm{can}"] & (\mathcal{F}_4)_{\{0\}} \arrow[r] & 0,
\end{tikzcd}$$

noting that its first line is isomorphic to the sequence (A.4) after applying $(\bullet)_{S_{41} \cup S_{34}}$. Note also that the condition $\sigma_4 \sigma_3 \sigma_2 \sigma_1 = \mathrm{id}$ is necessary here for the diagram to commute.

Now, define $\mathcal{F}$ by the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{123} \oplus \mathcal{F}_4 \xrightarrow{\gamma \circ \mathrm{can} - \mathrm{can}} (\mathcal{F}_4)_{S_{41} \cup S_{34}} \longrightarrow 0.$$

(4) Construct the isomorphisms α_k:
We have already constructed isomorphisms $(\mathcal{F}_{12})_{S_1} \simeq \mathcal{F}_1$ and $(\mathcal{F}_{12})_{S_2} \simeq \mathcal{F}_2$ above. Starting from these, one can in a similar manner obtain isomorphisms $(\mathcal{F}_{123})_{S_k} \simeq \mathcal{F}_k$ for $k \in \{1,2,3\}$ and finally $\mathcal{F}_{S_k} \simeq \mathcal{F}_k$ for $k \in \mathbb{Z}/4\mathbb{Z}$. As an exemplary case, let us explain the construction of the isomorphism $(\mathcal{F}_{123})_{S_1} \simeq \mathcal{F}_1$. Denote by $\beta \colon (\mathcal{F}_{12})_{S_1} \xrightarrow{\sim} \mathcal{F}_1$ the isomorphism constructed in diagram (A.2).

One can apply the functor $(\bullet)_{S_1}$ to (A.4) and there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F}_{123})_{S_1} \arrow[r] \arrow[d, dashed, "\simeq"] & (\mathcal{F}_{12})_{S_1} \oplus (\mathcal{F}_3)_{\{0\}} \arrow[r, "\xi \circ \mathrm{can} - \mathrm{id}"] \arrow[d, "\beta"', "\sigma_1^{-1} \circ \sigma_2^{-1}"] & (\mathcal{F}_3)_{\{0\}} \arrow[r] \arrow[d, "\sigma_1^{-1} \circ \sigma_2^{-1}"] & 0 \\
0 \arrow[r] & (\mathcal{F}_1) \arrow[r] & \mathcal{F}_1 \oplus (\mathcal{F}_1)_{\{0\}} \arrow[r, "\mathrm{can} - \mathrm{can}"] & (\mathcal{F}_1)_{\{0\}} \arrow[r] & 0,
\end{tikzcd}$$

We obtain the dashed isomorphism as soon as we have checked that the square on the right commutes, i.e. that $\mathrm{can} \circ \beta = \sigma_1^{-1} \circ \sigma_2^{-1} \circ \xi \circ \mathrm{can}$. To see this, consider the commutative diagram whose central part is induced by (A.3)

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F}_{12})_{S_1} \arrow[r] \arrow[d, "\mathrm{can}"] & \mathcal{F}_1 \oplus (\mathcal{F}_2)_{S_{12}} \arrow[r, "\sigma_1 \circ \mathrm{can} - \mathrm{id}"] \arrow[d, "\mathrm{can}" left, "\mathrm{can}"] & (\mathcal{F}_2)_{S_{12}} \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & (\mathcal{F}_{12})_{\{0\}} \arrow[r] \arrow[d, "\xi"] & (\mathcal{F}_1)_{\{0\}} \oplus (\mathcal{F}_2)_{\{0\}} \arrow[r, "\sigma_1 - \mathrm{id}"] \arrow[d, "\sigma_2 \circ \sigma_1" left, "\sigma_2"] & (\mathcal{F}_2)_{\{0\}} \arrow[r] \arrow[d, "\sigma_2"] & 0 \\
0 \arrow[r] & (\mathcal{F}_3)_{\{0\}} \arrow[r] \arrow[d, "\sigma_1^{-1}\sigma_2^{-1}"] & (\mathcal{F}_3)_{\{0\}} \oplus (\mathcal{F}_3)_{\{0\}} \arrow[r] \arrow[d, "\sigma_1^{-1}\sigma_2^{-1}" left, "\sigma_1^{-1}\sigma_2^{-1}"] & (\mathcal{F}_3)_{\{0\}} \arrow[r] \arrow[d, "\sigma_1^{-1}\sigma_2^{-1}"] & 0 \\
0 \arrow[r] & (\mathcal{F}_1)_{\{0\}} \arrow[r] & (\mathcal{F}_1)_{\{0\}} \oplus (\mathcal{F}_1)_{\{0\}} \arrow[r] & (\mathcal{F}_1)_{\{0\}} \arrow[r] & 0.
\end{tikzcd}$$

Consequently, the morphism $\sigma_1^{-1} \circ \sigma_2^{-1} \circ \xi \circ \mathrm{can}$ completes a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F}_{12})_{S_1} \arrow[r] & \mathcal{F}_1 \oplus (\mathcal{F}_2)_{S_{12}} \arrow[r, "\sigma_1 \circ \mathrm{can} - \mathrm{id}"] \arrow[d, "\sigma_1^{-1}" left, "\mathrm{can}"] & (\mathcal{F}_2)_{S_{12}} \arrow[r] \arrow[d, "\sigma_1^{-1}"] & 0 \\
0 \arrow[r] & (\mathcal{F}_1)_{\{0\}} \arrow[r] & (\mathcal{F}_1)_{\{0\}} \oplus (\mathcal{F}_1)_{\{0\}} \arrow[r] & (\mathcal{F}_1)_{\{0\}} \arrow[r] & 0,
\end{tikzcd}$$

and so does $\mathrm{can} \circ \beta$ (this is clear considering (A.2)). Since such a morphism is unique, equality follows.

Once all the $\mathcal{F}_{S_k} \simeq \mathcal{F}_k$ have been determined, one can check that the induced transition morphisms are again the σ_k from the data we started with.

(5) Uniqueness:
It remains to show that $\mathcal{F}$ is unique up to unique isomorphism. Assume that $\mathcal{F}'$ is another sheaf satisfying the properties of being isomorphic to $\mathcal{F}_k$ on S_k with transition morphisms given by the σ_k. Then one can construct an isomorphism $\mathcal{F} \simeq \mathcal{F}'$ from the given isomorphisms $\mathcal{F}_{S_k} \simeq \mathcal{F}_k \simeq \mathcal{F}'_{S_k}$. This is again done by means of morphisms of short exact sequences. For example, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{S_1 \cup S_2} \arrow[r] \arrow[d, dashed, "\simeq"] & \mathcal{F}_{S_1} \oplus \mathcal{F}_{S_2} \arrow[r] \arrow[d, "\simeq" left, "\simeq"] & \mathcal{F}_{S_{12}} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathcal{F}'_{S_1 \cup S_2} \arrow[r] & \mathcal{F}'_{S_1} \oplus \mathcal{F}'_{S_2} \arrow[r] & \mathcal{F}'_{S_{12}} \arrow[r] & 0
\end{tikzcd}$$

produces an isomorphism on $S_1 \cup S_2$ extending those on S_1 and S_2. In this manner, one finally obtains the global isomorphism.

This isomorphism is unique (with the property that it extends the given isomorphisms on the S_k) since it induces isomorphisms of the short exact sequences involved in the construction, and there the dashed isomorphism is unique.

□

Note that the lemma holds completely analogously if we drop the condition $|z| \leq R$ in the definition of S_k (i.e. if we consider sectors of infinite radius), if we do not assume the sectors to be right-angled, or if we assume the $\mathcal{F}_k$ to be sheaves on $S_k \times \mathbb{R}$ rather than sheaves on S_k (as we do in our considerations with enhanced sheaves).

Appendix B.

Topology of intersections

In this appendix, we give the details on the study of the topological spaces with which we were concerned in Section 3.3.3 and Section 3.4.2.

B.1. Cohomology with compact support

Let us first fix some notation: For a topological space X, we will denote by $\mathrm{H}_l(X)$ (resp. $\mathrm{H}^l(X)$) its singular homology (resp. cohomology) groups. The singular homology and cohomology with coefficients in $\mathbf{k}$ will be denoted by $\mathrm{H}_l(X;\mathbf{k})$ and $\mathrm{H}^l(X;\mathbf{k})$, respectively. Similarly, relative homology of a pair (X,A) is denoted by $\mathrm{H}_l(X,A)$ and $\mathrm{H}_l(X,A;\mathbf{k})$, and accordingly for cohomology. The cohomology with compact support can be defined by

$$\mathrm{H}^l_{\mathrm{c}}(X;\mathbf{k}) := \varinjlim_{\substack{K\subseteq X\\ K \text{ compact}}} \mathrm{H}^l(X, X\smallsetminus K;\mathbf{k}).$$

If X is compact, this implies $\mathrm{H}^l_{\mathrm{c}}(X;\mathbf{k}) \simeq \mathrm{H}^l(X;\mathbf{k})$.

It is well-known that the cohomology sends disjoint unions to products, and in particular for finite disjoint unions one has

$$\mathrm{H}^l_{\mathrm{c}}(X\sqcup Y;\mathbf{k}) \simeq \mathrm{H}^l_{\mathrm{c}}(X;\mathbf{k}) \oplus \mathrm{H}^l_{\mathrm{c}}(Y;\mathbf{k}).$$

It is important to keep in mind that in general cohomology with compact support is *not* invariant under homotopy equivalences (in contrast to ordinary cohomology), but only under homeomorphisms.

The first lemma we need is a statement about the compactly supported cohomology of certain regions between two graphs of functions.

Lemma B.1. *Let $I\subseteq\mathbb{R}$ be a closed interval and let $f,g\colon I\to\mathbb{R}$ be continuous functions with $g(x)\geq f(x)$ for any $x\in I$. Set $G:=\{(x,y)\in\mathbb{R}^2 \mid x\in I, f(x)\leq y\leq g(x)\}\subseteq\mathbb{R}^2$. Then the following holds:*

(i) If $I=[a,\infty)$ for some $a\in\mathbb{R}$, one has

$$\mathrm{H}^l_{\mathrm{c}}(G;\mathbf{k})\simeq 0$$

for any $l\in\mathbb{Z}$.

(ii) If $I = \mathbb{R}$, one has

$$\mathrm{H}^l_{\mathrm{c}}(G; \mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $I = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$ and $f(x) < g(x)$ for $x \in (a, b)$, one has

$$\mathrm{H}^l_{\mathrm{c}}(G \smallsetminus \{p\}; \mathbf{k}) \simeq 0$$

for any $l \in \mathbb{Z}$, where $p = (b, f(b))$.

Now let $I \subseteq \mathbb{R}$ be any (open or closed, bounded or unbounded) interval and let $d \in \mathbb{R} \sqcup \{\infty\}$ with $f(x) < d$ for any $x \in I$. Set $E := \{(x, y) \in \mathbb{R}^2 \mid x \in I, f(x) \leq y < d\} \subseteq \mathbb{R}^2$.

(iv) One has

$$\mathrm{H}^l_{\mathrm{c}}(E; \mathbf{k}) \simeq 0$$

for any $l \in \mathbb{Z}$.

Proof. By definition of compactly supported cohomology and the universal coefficient theorem, one has

$$\begin{aligned} \mathrm{H}^l_{\mathrm{c}}(X; \mathbf{k}) &\simeq \varinjlim_{\substack{K \subseteq X \\ K \text{ compact}}} \mathrm{H}^l(X, X \smallsetminus K; \mathbf{k}) \\ &\simeq \varinjlim_{\substack{K \subseteq X \\ K \text{ compact}}} \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{H}_l(X, X \smallsetminus K), \mathbf{k}\big). \end{aligned} \tag{B.1}$$

We first prove (i). It suffices to consider compact sets of the form $C = ([a, b] \times \mathbb{R}) \cap G$ since any compact subset of G is contained in one of this form. Precisely, if $\mathrm{pr}_1 \colon \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first component and $K \subset G$ is compact, set $b := \max\{\mathrm{pr}_1(p) \mid p \in K\}$. Note that C is indeed bounded and hence compact because f has a minimum and g has a maximum on the closed interval $[a, b]$.

In order to compute $\mathrm{H}_l(G, G \smallsetminus C)$, we apply the long exact sequence in homology

$$\begin{aligned} \ldots \to \overbrace{\mathrm{H}_2(G)}^{=0} \to \mathrm{H}_2(G, G \smallsetminus C) \to \overbrace{\mathrm{H}_1(G \smallsetminus C)}^{=0} \to \overbrace{\mathrm{H}_1(G)}^{=0} \to \mathrm{H}_1(G, G \smallsetminus C) \\ \to \underbrace{\mathrm{H}_0(G \smallsetminus C)}_{=\mathbb{Z}} \to \underbrace{\mathrm{H}_0(G)}_{=\mathbb{Z}} \to \mathrm{H}_0(G, G \smallsetminus C) \to 0 \end{aligned}$$

and the fact that both G and $G \smallsetminus C$ are contractible (they are clearly homotopy equivalent to a half-line), i.e. they have non-vanishing homologies only in degree 0 and the map $\mathbb{Z} \to \mathbb{Z}$ in the long exact sequence above is the identity. We conclude that $\mathrm{H}_l(G, G \smallsetminus C) \simeq 0$ for any l. Hence, by (B.1), the assertion follows.

The proof of (ii) is similar. Here, we can consider compact sets of the form $C = ([-b, b] \times \mathbb{R}) \cap G$, since any compact $K \subset G$ is contained in one of this form, setting

$b := \max\{|\mathrm{pr}_1(p)| \mid p \in K\}$. Again, G is contractible. The space $G \smallsetminus C$ consists of two contractible connected components and hence the long exact sequence is

$$\begin{aligned} \ldots \to \overbrace{\mathrm{H}_2(G)}^{=0} \to \mathrm{H}_2(G, G \smallsetminus C) \to \overbrace{\mathrm{H}_1(G \smallsetminus C)}^{=0} \to \overbrace{\mathrm{H}_1(G)}^{=0} \to \mathrm{H}_1(G, G \smallsetminus C) \\ \to \underbrace{\mathrm{H}_0(G \smallsetminus C)}_{=\mathbb{Z}^2} \to \underbrace{\mathrm{H}_0(G)}_{=\mathbb{Z}} \to \mathrm{H}_0(G, G \smallsetminus C) \to 0, \end{aligned}$$

where the map $\mathbb{Z}^2 \to \mathbb{Z}$ is given by addition. Therefore, we get $\mathrm{H}_1(G, G \smallsetminus C) \simeq \mathbb{Z}$ and $\mathrm{H}_l(G, G \smallsetminus C) \simeq 0$ for any $l \neq 1$ and any C of the form above. Then the assertion follows.

We prove (iii) in the same manner: Note first that G is homeomorphic to a closed square or a closed triangle (depending on whether $f(a) = g(a)$ and $f(b) = g(b)$), hence contractible (and so is $G \smallsetminus \{p\}$). Let us assume that G is such a square or triangle. Any compact set $K \subset G \smallsetminus \{p\}$ is contained in one of the form $C = G \smallsetminus B_p$ where B_p is an open ball of sufficiently small radius around p. Then $G \smallsetminus C$ is still star-shaped, hence contractible and the long exact sequence in homology is as in (i), with G replaced by $G \smallsetminus \{p\}$.

Finally, to prove (iv), it suffices to consider $C = ([-b, b] \times (-\infty, c]) \cap E$ with $b := \max\{|\mathrm{pr}_1(p)| \mid p \in K\}$ and $c := \max\{\mathrm{pr}_2(p) \mid p \in K\}$ for any compact $K \subset E$. Here, $\mathrm{pr}_2 \colon \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the second component. Note that $c < d$. Since E and $E \smallsetminus C$ are homotopy equivalent to a rectangle $\{(x, y) \in \mathbb{R}^2 \mid x \in I, s \leq y < d\} \in \mathbb{R}^2$ (for any $s < d$) and hence contractible, the long exact sequence in homology is the same as for (i), with G replaced by E. This concludes the proof. $\square$

Compactly supported cohomology is also related to the proper direct image functor. Let X be a good topological space and $\mathrm{a}_X \colon X \to \{\mathrm{pt}\}$ the unique map to the one-point space. For $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_X)$, one defines cohomology with compact support with values in $\mathcal{F}$ as

$$\mathrm{H}^l_{\mathrm{c}}(X; \mathcal{F}) := \mathrm{R}^l \mathrm{a}_{X!} \mathcal{F}.$$

We have the following well-known results.

Lemma B.2. *Let $f \colon X \to Y$ be a morphism of good topological spaces and $l \in \mathbb{Z}$.*

(i) ([21, Proposition 2.5.2]) Let $\mathcal{F} \in \mathrm{Mod}(\mathbf{k}_X)$. For any $y \in Y$, there is an isomorphism

$$(\mathrm{R}^l f_! \mathcal{F})_y \simeq \mathrm{H}^l_{\mathrm{c}}\big(f^{-1}(y); \mathcal{F}|_{f^{-1}(y)}\big).$$

(ii) Let $j \colon X' \to X$ be an injective continuous map and $\mathcal{G} \in \mathrm{Mod}(\mathbf{k}_{X'})$. Then

$$\mathrm{H}^l_{\mathrm{c}}(X; j_! \mathcal{G}) \simeq \mathrm{H}^l_{\mathrm{c}}(X'; \mathcal{G}).$$

(iii) Let X be a locally contractible space. Then cohomology with values in the constant sheaf is isomorphic to the usual (singular) cohomology, i.e.

$$\mathrm{H}^l_{\mathrm{c}}(X; \mathbf{k}_X) \simeq \mathrm{H}^l_{\mathrm{c}}(X; \mathbf{k}).$$

Proof. Statement (ii) is an easy consequence of the definition: Denote by $\mathrm{a}_X \colon X \to \{\mathrm{pt}\}$ and $\mathrm{a}_{X'} \colon X' \to \{\mathrm{pt}\}$ the unique maps. It is clear that $\mathrm{a}_{X'} = \mathrm{a}_X \circ j$. Hence

$$\mathrm{H}^l_{\mathrm{c}}(X; j_! \mathcal{G}) \simeq \mathrm{R}^l \mathrm{a}_{X!} j_! \mathcal{G} \simeq \mathrm{R}^l \mathrm{a}_{X'!} \mathcal{G} \simeq \mathrm{H}^l_{\mathrm{c}}(X'; \mathcal{G}).$$

Part (iii) follows from the classical fact that for locally contractible spaces singular cohomology agrees with sheaf cohomology of the constant sheaf, and this holds also in the case of cohomology with appropriate (e.g. compact) supports (see [2, Chapter III], for example). $\square$

B.2. Hyperbolae and right-angled sectors

In this and the following section, we investigate the intersections of hyperbolic regions and sectors occuring in our computations of the Fourier–Laplace transform. Let us recall that a hyperbola in standard form is described by an equation

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$$

for positive real numbers a and b. It is not difficult to determine vertices and asymptotes: The vertices of the hyperbola are at the points $(\pm a, 0)$. The asymptotes are lines through the origin of slope $\pm\frac{b}{a}$.

B.2.1. Hyperbolae and sectors

In Section 3.3.3, an important step is the determination of the cohomology with compact support of a topological space $\mathcal{X} \subseteq \mathbb{R}^2$ which is given as the intersection of the hyperbolic region defined by

$$\frac{c_1}{2} x_1^2 - \frac{|c|^2}{2c_1} x_2^2 \leq \check{t} + \operatorname{Re} \frac{1}{2c} \check{w}^2 \tag{B.2}$$

and the right-angled sector given by

$$x_1 \geq \frac{\check{w}_1}{c_1}, \quad x_2 \geq \frac{c_1 \check{w}_2 - c_2 \check{w}_1}{|c|^2}$$

where $c = c_1 + ic_2 \in \mathbb{C}$ with $c_1 > 0$ is fixed and $\check{w} = \check{w}_1 + i\check{w}_2 \in \mathbb{C}$, $\check{t} \in \mathbb{R}$ are parameters.

The parameters $\check{w}$ and $\check{t}$ influence the exact shape of the hyperbolic region as well as the relative position of hyperbola and sector. Since the geometry of the hyperbolic region highly depends on the sign of the right-hand side of (B.2), we distinguish two cases. Let us write for short $\kappa(\check{w}, \check{t}) := \check{t} + \operatorname{Re} \frac{1}{2c} \check{w}^2$.

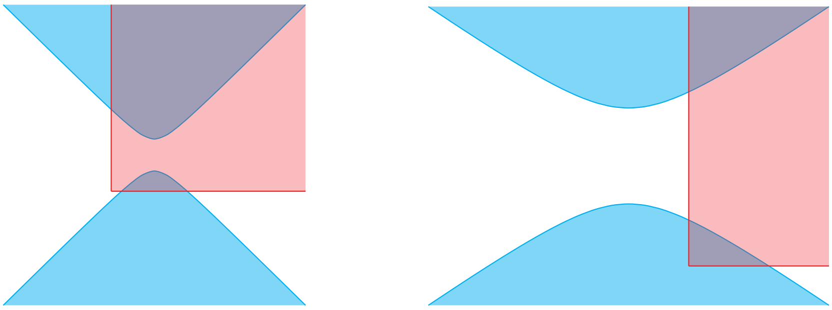


Figure B.1.: The case $\kappa(\check{w},\check{t}) < 0$: A sketch of the two situations in which the intersection of hyperbolic region and sector can have a compact connected component.

Case 1: $\kappa(\check{w},\check{t}) < 0$

In this case, (B.2) can be written as

$$\frac{|c|^2}{-2c_1\kappa(\check{w},\check{t})}x_2^2 - \frac{c_1}{-2\kappa(\check{w},\check{t})}x_1^2 \geq 1,$$

which describes the region outside the two branches of a hyperbola and therefore consists of two connected components (see Fig. 3.1 (a), p. 53). Independently of the position of the sector, its intersection with the upper component of the hyperbolic region will always be an epigraph of the form introduced in Lemma B.1 (iv), and hence its compactly supported cohomology vanishes. If the sector intersects only the upper branch of the hyperbola, we therefore have $\mathrm{H}^l_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq 0$ for any l.

If the sector intersects in addition the lower branch of the hyperbola, $\mathcal{X}$ has a second connected component, and the latter is compact and convex, hence contractible. Therefore, writing $\mathcal{X} = E \sqcup C$ with E the epigraph and C the compact component, we have $\mathrm{H}^l_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq \mathrm{H}^l_{\mathrm{c}}(C;\mathbf{k}) \simeq \mathrm{H}^l(C;\mathbf{k})$, so $\mathrm{H}^0_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq \mathbf{k}$ and $\mathrm{H}^l_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq 0$ for $l \neq 0$.

The conditions under which an intersection of the sector with the lower branch of the hyperbola occurs can be formulated as follows (see also Fig. B.1): Either the sector's corner is in the left half-plane and at least as low as the lower vertex of the hyperbola, in symbols:

$$\frac{\check{w}_1}{c_1} \leq 0 \quad \text{and} \quad \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \leq -\sqrt{\frac{-2c_1\kappa(\check{w},\check{t})}{|c|^2}},$$

or the corner of the sector is in the right half-plane and is low enough for the sector to intersect the lower branch of the hyperbola, in symbols:

$$\frac{\check{w}_1}{c_1} > 0 \quad \text{and} \quad \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \leq -\sqrt{\frac{-2c_1\kappa(\check{w},\check{t})}{|c|^2}\left(1 + \frac{c_1}{-2\kappa(\check{w},\check{t})}\left(\frac{\check{w}_1}{c_1}\right)^2\right)}.$$

Doing some arithmetic, these conditions can be rewritten as follows: In the case $\check{t} <$

$-\operatorname{Re}\frac{1}{2c}\check{w}^2$, there is a compact connected component in $\mathcal{X}$ if and only if

$$\check{w}_1 \leq 0 \quad \text{and} \quad c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq -\frac{\check{w}_1^2}{2c_1}$$

or

$$w_1 > 0 \quad \text{and} \quad c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq 0.$$

Case 2: $\kappa(\check{w}, \check{t}) \geq 0$
Here, (B.2) reads as

$$\frac{c_1}{2}x_1^2 - \frac{|c|^2}{2c_1}x_2^2 \leq \kappa(\check{w}, \check{t})$$

and describes the region between the two branches of a (possibly degenerate) hyperbola (see Fig. 3.1 (b) and (c), p. 53).

Again, the intersection of hyperbolic region and sector can consist of one (unbounded) or two (one unbounded and one compact) connected component. The unbounded component is (up to rotation by 90 degrees) always of the form treated in Lemma B.1 (i) and hence does not contribute to the compactly supported cohomology of $\mathcal{X}$. On the other hand, the compact component is contractible and hence (in case it exists) $\mathrm{H}^0_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq \mathbf{k}$ and $\mathrm{H}^l_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq 0$ for $l \neq 1$.

Such a compact connected component exists if and only if the following is the case (cf. Fig. B.2): The corner of the sector lies to the right of the hyperbola's right vertex and low enough such that the sector intersects the lower part of the hyperbola's right branch, in symbols:

$$\frac{\check{w}_1}{c_1} > \sqrt{\frac{2\kappa(\check{w},\check{t})}{c_1}} \quad \text{and} \quad \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \leq -\sqrt{\frac{2c_1\kappa(\check{w},\check{t})}{|c|^2}\left(\frac{c_1}{2\kappa(\check{w},\check{t})}\left(\frac{\check{w}_1}{c_1}\right)^2 - 1\right)}$$

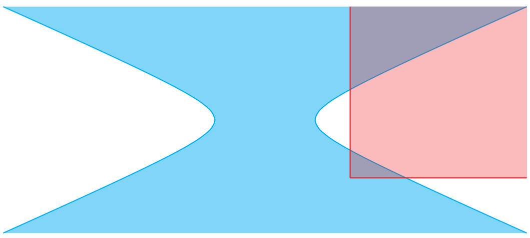


Figure B.2.: The case $\kappa(\check{w}, \check{t}) \geq 0$: A sketch of a situation in which the intersection of hyperbolic region and sector has a compact connected component.

This can be shown to be equivalent to

$$\check{w}_1 > 0 \quad \text{and} \quad \check{t} < \frac{1}{2c_1|c|^2}(c_1\check{w}_2 - c_2\check{w}_1)^2 \quad \text{and} \quad c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq 0,$$

which is therefore the condition for the existence of a compact connected component in $\mathcal{X}$ in the case $\check{t} \geq -\operatorname{Re}\frac{1}{2c}\check{w}^2$.

We will write for short

$$\eta_c(w) := -\frac{1}{2c_1|c|^2}(c_1 w_2 - c_2 w_1)^2$$

for $w \in \mathbb{C}$, and we note that $-\eta_c(w) \geq -\operatorname{Re}\frac{1}{2c}w^2 \geq -\frac{w_1^2}{2c_1}$ since $-\eta_c(w) + \operatorname{Re}\frac{1}{2c}w^2 = \frac{w_1^2}{2c_1}$.

Putting the two cases together, we get the following result for the cohomology with compact support of $\mathcal{X}$:

$$\mathrm{H}^l_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq 0 \quad \text{if } l \neq 0$$

and

$$\mathrm{H}^0_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \text{ and } -\varphi^+_{\mathrm{r},c}(\check{w}) \leq \check{t} < -\varphi^-_{\mathrm{r},c}(\check{w}), \\ 0, & \text{otherwise,} \end{cases}$$

where the functions $\varphi^+_{\mathrm{r},c}, \varphi^-_{\mathrm{r},c} \colon \mathbb{C} \to \mathbb{R}$ are defined by

$$\varphi^+_{\mathrm{r},c}(w) := \begin{cases} \frac{w_1^2}{2c_1} & \text{if } w_1 \leq 0, \\ 0 & \text{if } w_1 > 0 \end{cases}$$

and

$$\varphi^-_{\mathrm{r},c}(w) := \begin{cases} \operatorname{Re}\frac{1}{2c}w^2 & \text{if } w_1 \leq 0, \\ \eta_c(w) & \text{if } w_1 > 0. \end{cases}$$

One easily verifies that these functions are continuous.

Considerations with a sector which is not a translation of the first quadrant (as in the cases just discussed) but of the second, third and fourth quadrant are completely analogous and yield similar results.

B.2.2. Hyperbolae and half-lines

Another important region coming from Section 3.3.3 is the intersection of the hyperbolic region with horizontal and vertical half-lines (i.e. with the borders between the sectors considered above). The determination of its compactly supported cohomology is analogous to the previous section.

Horizontal half-line Let us consider the space defined as the intersection $\mathcal{Y}$ of the hyperbolic region given by (as above)

$$\frac{c_1}{2}x_1^2 - \frac{|c|^2}{2c_1}x_2^2 \leq \kappa(\check{w}, \check{t})$$

and the horizontal half-line given by

$$x_1 \geq \frac{\check{w}_1}{c_1}, \quad x_2 = \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2}.$$

As soon as this intersection is nonempty, $\mathcal{Y}$ is compact and contractible. The conditions for a nonempty intersection are as follows:

If $\kappa(\check{w}, \check{t}) < 0$, the absolute value of x_2 must be big enough in order to intersect the upper or lower branch of the hyperbola. More precisely, if $\check{w}_1 \leq 0$, it must be at least the absolute value of the hyperbola's vertices' distance from the origin, i.e.

$$\check{w}_1 \leq 0 \quad \text{and} \quad \left| \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \right| \geq \sqrt{\frac{-2c_1\kappa(\check{w}, \check{t})}{|c|^2}},$$

or, if $\check{w}_1 > 0$, it must be at least as high as the points on the hyperbola with $x_1 = \frac{\check{w}_1}{2c_1}$, i.e.

$$\check{w}_1 > 0 \quad \text{and} \quad \left| \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \right| \geq \sqrt{\frac{-2c_1\kappa(\check{w}, \check{t})}{|c|^2} \left(1 + \frac{c_1}{-2\kappa(\check{w}, \check{t})} \left(\frac{\check{w}_1}{c_1} \right)^2 \right)}.$$

These conditions are equivalent to asking that

$$\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq -\frac{\check{w}_1^2}{2c_1}$$

or

$$\check{w}_1 > 0 \quad \text{and} \quad \check{t} \geq 0.$$

If $\kappa(\check{w}, \check{t}) \geq 0$, the intersection is nonempty if and only if the half-line intersects the right branch of the hyperbola, i.e. if and only if

$$\frac{\check{w}_1}{c_1} \leq \sqrt{\frac{2\kappa(\check{w}, \check{t})}{c_1} \left(1 + \frac{|c|^2}{2c_1\kappa(\check{w}, \check{t})} \left(\frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \right)^2 \right)}.$$

(The right-hand side is the x_1-value of the point on the right branch with $x_2 = \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2}$.) This condition is equivalent to asking that either $\check{w}_1 \leq 0$ or

$$\check{w}_1 > 0 \quad \text{and} \quad \check{t} \geq 0.$$

In conclusion, we obtain

$$\mathrm{H}^l_{\mathrm{c}}(\mathcal{Y}; \mathbf{k}) \simeq 0 \quad \text{if } l \neq 0$$

and

$$\mathrm{H}^0_{\mathrm{c}}(\mathcal{Y}; \mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } \check{t} \geq -\varphi^+_{\mathrm{r},c}(\check{w}), \\ 0 & \text{otherwise.} \end{cases}$$

Vertical half-line Let us now compute the compactly supported cohomology of the space defined as the intersection $\mathcal{Y}$ of the hyperbolic region given by (as above)

$$\frac{c_1}{2}x_1^2 - \frac{|c|^2}{2c_1}x_2^2 \leq \kappa(\check{w}, \check{t})$$

and the vertical half-line given by

$$x_1 = \frac{\check{w}_1}{c_1}, \quad x_2 \geq \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2}.$$

This intersection may consist of one (unbounded) or two (one unbounded and one compact contractible) connected component. The unbounded component is of the form as in Lemma B.1 (i) and therefore does not contribute to the compactly supported cohomology. The conditions for the existence of a compact connected component in the intersection are as follows:

If $\kappa(\check{w}, \check{t}) < 0$, the starting point of the half-line must be low enough to intersect the hyperbola's lower branch, i.e.

$$\frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \leq -\sqrt{\frac{2c_1\kappa(\check{w}, \check{t})}{|c|^2}\left(\frac{c_1}{2\kappa(\check{w}, \check{t})}\left(\frac{\check{w}_1}{c_1}\right)^2 - 1\right)},$$

and this condition is equivalent to asking that

$$c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq 0.$$

If $\kappa(\check{w}, \check{t}) \geq 0$, the starting point of the half-line must have an x_1-value whose absolute value is greater than the one of the hyperbola's vertices, and the x_2-value must be small enough to intersect the lower part of one of the branches, i.e.

$$\left|\frac{\check{w}_1}{c_1}\right| > \sqrt{\frac{2\kappa(\check{w}, \check{t})}{c_1}} \quad \text{and} \quad \frac{c_1\check{w}_2 - c_2\check{w}_1}{|c|^2} \leq -\sqrt{\frac{2c_1\kappa(\check{w}, \check{t})}{|c|^2}\left(\frac{c_1}{2\kappa(\check{w}, \check{t})}\left(\frac{\check{w}_1}{c_1}\right)^2 - 1\right)},$$

and this is equivalent to

$$\check{t} < -\eta_c(\check{w}) \quad \text{and} \quad c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq 0.$$

(Note that we do not have to ask $\check{w}_1 \neq 0$ separately since $\operatorname{Re}\frac{1}{2c}\check{w}$ and $\eta_c(\check{w})$ coincide for $\check{w}_1 = 0$ and hence there are no possible values for $\check{t}$.)

We conclude that

$$\mathrm{H}^l_\mathrm{c}(\mathcal{Y}; \mathbf{k}) \simeq 0 \quad \text{if } l \neq 0$$

and

$$\mathrm{H}^0_\mathrm{c}(\mathcal{Y}; \mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \text{ and } 0 \leq \check{t} < -\eta_c(\check{w}), \\ 0 & \text{otherwise.} \end{cases}$$

B.3. Hyperbolae and more general sectors

B.3.1. Hyperbolae and sectors

In Section 3.4.2, we work with two fixed parameters $c = c_1 + ic_2, d = d_1 + id_2 \in \mathbb{C}^\times$ with $c_1 > 0$, $c_2 \geq 0$, $d_1 > c_1$ and $c_1 d_2 - c_2 d_1 \geq 0$. The main step there is to compute the cohomology with compact support of the space $\mathcal{X}$ defined as the intersection of the hyperbolic region given by

$$\frac{d_1}{2} x_1^2 - \frac{|d|^2}{2d_1} x_2^2 \leq \check{t} + \operatorname{Re} \frac{1}{2d} \check{w}^2 \tag{B.3}$$

and the sector defined by

$$(c_1 d_2 - c_2 d_1)\left(x_2 - \frac{d_1 \check{w}_2 - d_2 \check{w}_1}{|d|^2}\right) \geq -c_1 d_1 \left(x_1 - \frac{\check{w}_1}{d_1}\right), \quad x_2 \geq \frac{d_1 \check{w}_2 - d_2 \check{w}_1}{|d|^2}. \tag{B.4}$$

Again, $\check{w} = \check{w}_1 + i\check{w}_2 \in \mathbb{C}$ and $\check{t} \in \mathbb{R}$ are parameters for this space. Note that this includes the case of Section B.2 for $c_1 d_2 - c_2 d_1 = 0$.

The sector described by (B.4) has its corner at the point $(\frac{\check{w}_1}{d_1}, \frac{d_1 \check{w}_2 - d_2 \check{w}_1}{|d|^2})$, and its borders are a horizontal half-line to the right and a half-line with (negative) slope $m := -\frac{c_1 d_1}{c_1 d_2 - c_2 d_1}$ (or a vertical half-line if $c_1 d_2 - c_2 d_1 = 0$) unbounded from above.

The region (B.3) is bounded by a hyperbola centered at the origin with asymptotes of slope $\mu_\pm := \pm \frac{d_1}{|d|}$. An easy algebraic manipulation shows that

$$m^2 - \mu_-^2 = \frac{c_1 d_1^3 (c_1 d_1 + c_2 d_2) + c_2 d_1^3 (c_1 d_2 - c_2 d_1)}{(c_1 d_2 - c_2 d_1)^2 |d|^2} > 0,$$

so the non-horizontal border of the sector is steeper than the hyperbola's decreasing asymptote.

In order to determine the conditions on $\check{w}$ and $\check{t}$ for nontrivial compactly supported cohomology of $\mathcal{X}$, we distinguish two cases, since the geometry of the hyperbolic region highly depends on the right-hand side of (B.3).

Case 1: $\check{t} < -\operatorname{Re} \frac{1}{2d} \check{w}^2$

In this case, the unbounded connected component of $\mathcal{X}$ is always an epigraph as described in Lemma B.1 (iv) and as such does not contribute to $\mathrm{H}^l_c(\mathcal{X}; \mathbf{k})$. If a nonempty second connected component exists, it is compact and contractible and therefore produces a nontrivial first cohomology group with compact support. Similarly to the considerations in Section B.2.1, this happens if and only if the sector is low enough such that it intersects the lower branch of the hyperbola. One finds that the conditions for this are

$$\check{w}_1 \leq 0 \quad \text{and} \quad d_1 \check{w}_2 - c_2 \check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq -\frac{\check{w}_1^2}{2d_1}$$

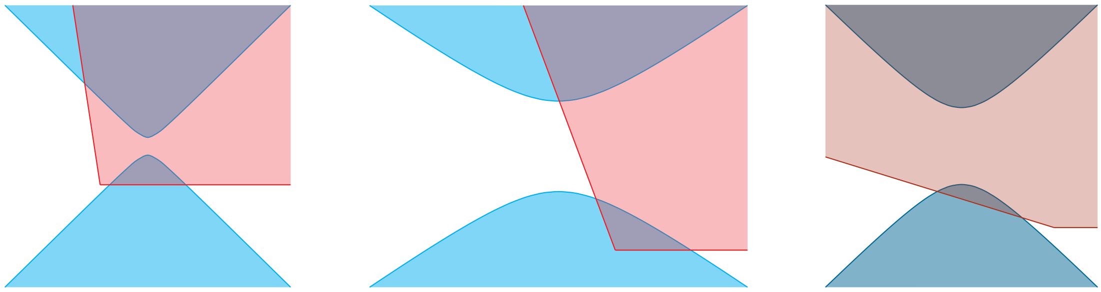


Figure B.3.: The case $\check{t} < -\operatorname{Re} \frac{1}{2d} \check{w}^2$: A sketch of the two situations in which the intersection of hyperbolic region and sector can have a compact connected component. The third situation is impossible due to the slope of the half-line bounding the sector.

or

$$\check{w}_1 > 0 \quad \text{and} \quad d_1 \check{w}_2 - d_2 \check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq 0.$$

A situation as shown in the rightmost picture of Fig. B.3 cannot occur since the hyperbola is never as steep as the sector's border.

Case 2: $\check{t} \geq -\operatorname{Re} \frac{1}{2d} \check{w}^2$
This case is a bit more complicated: The condition under which a compact connected component of $\mathcal{X}$ exists is as follows (cf. Fig. B.4):

The non-horizontal border of the sector must intersect
the right branch of the hyperbola in two distinct points.

Making this more explicit requires a little effort: The right branch of the hyperbola is described by

$$x_1 = \sqrt{\frac{|d|^2}{d_1^2} x_2^2 + \frac{2}{d_1} \left(\check{t} + \operatorname{Re} \frac{1}{2d} \check{w}^2 \right)},$$

and the border of the sector is the part of the line

$$x_1 = -\frac{c_1 d_2 - c_2 d_1}{c_1 d_1} \left(x_2 - \frac{d_1 \check{w}_2 - d_2 \check{w}_1}{|d|^2} \right) + \frac{\check{w}_1}{d_1},$$

where $x_1 \leq \frac{\check{w}_1}{d_1}$. Equating the two right-hand sides and taking the square yields a quadratic equation for x_2 whose discriminant is positive whenever

$$t < \frac{(c_1 \check{w}_2 - c_2 \check{w}_1)^2}{2(c_1^2 d_1 - c_2^2 d_1 + 2 c_1 c_2 d_2)}. \tag{B.5}$$

The two solutions for the values of x_2 and the corresponding values $x_1^{\pm}$ can then explicitly be computed using the quadratic formula, and we are interested in the conditions under

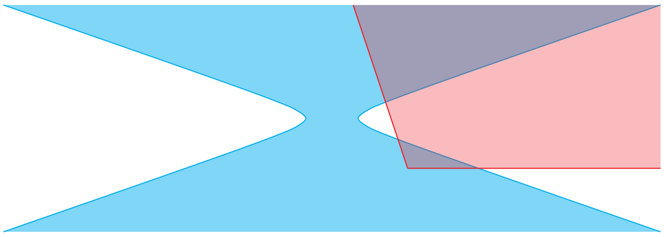


Figure B.4.: The case $\check{t} \geq -\operatorname{Re}\frac{1}{2d}\check{w}^2$: The intersection of hyperbolic region and sector has a compact connected component if and only if the sector intersects the lower part of the right branch in such a way that the intersection is disconnected (with one compact and one unbounded component).

which $0 < x_1^{\pm} \leq \frac{\check{w}_1}{d_1}$. (In particular, this is only possible if $\check{w}_1 > 0$.) This is the case if and only if, in addition to the above, we require

$$(c_1d_2 - c_2d_1)\check{w}_2 > -(c_1d_1 + c_2d_2)\check{w}_1 \quad \text{and} \quad \check{t} > -\frac{|c|^2 d_1}{2(c_1d_2 - c_2d_1)}\check{w}_1^2 - \frac{c_1}{c_1d_2 - c_2d_1}\check{w}_1\check{w}_2 \tag{B.6}$$

and

$$c_1\check{w}_2 - c_2\check{w}_1 \leq 0 \quad \text{and} \quad \check{t} \geq 0. \tag{B.7}$$

The conditions for the existence of a compact connected component in $\mathcal{X}$ in the case $\check{t} \geq -\operatorname{Re}\frac{1}{2d}\check{w}^2$ are therefore given by $\check{w}_1 > 0$ and (B.5)–(B.7).

The second condition in (B.6) is, however, obsolete, as we will show now: We subtract the right-hand side from $-\operatorname{Re}\frac{1}{2d}\check{w}^2 = -\frac{1}{2|d|^2}(d_1\check{w}_1^2 - d_1\check{w}_2^2 + 2d_2\check{w}_1\check{w}_2)$ and, after expanding and factoring, obtain

$$\begin{aligned}-\operatorname{Re}\frac{1}{2d}\check{w}^2 + \frac{|c|^2 d_1}{2(c_1d_2 - c_2d_1)}\check{w}_1^2 + \frac{c_1}{c_1d_2 - c_2d_1}\check{w}_1\check{w}_2 \\ = \frac{d_1\big((c_1d_1 + c_2d_2)\check{w}_1 + (c_1d_2 - c_2d_1)\check{w}_2\big)^2}{2|d|^2(c_1d_2 - c_2d_1)^2}.\end{aligned}$$

This is clearly non-negative. If $(c_1d_1 + c_2d_2)\check{w}_1 \neq -(c_1d_2 - c_2d_1)\check{w}_2$, it is even strictly positive, and hence the second condition in (B.6) can be neglected in view of the first and the global assumption $\check{t} \geq -\operatorname{Re}\frac{1}{2d}\check{w}^2$.

In order to put the two cases together, it is useful to know the following. We will abbreviate

$$\zeta(w) := -\frac{(c_1w_2 - c_2w_1)^2}{2(c_1^2d_1 - c_2^2d_1 + 2c_1c_2d_2)},$$

so the right-hand side of (B.5) is $-\zeta(\check{w})$.

Lemma B.3. *The following comparisons hold:*

(i) $-\operatorname{Re}\frac{1}{2d}\check{w}^2 \leq 0$ *if* $\check{w}_1 > 0$ *and* $\frac{c_2}{c_1}\check{w}_1 < \check{w}_2 \leq \frac{d_2}{d_1}\check{w}_1$,

(ii) $-\operatorname{Re}\frac{1}{2d}\check{w}^2 \leq -\zeta(\check{w})$ *for any* $\check{w} \in \mathbb{C}$.

Proof. (i) First, note that the condition required is only possible if $d_2 \neq 0$ (since $d_2 = 0$ would also imply $c_2 = 0$).

One obtains

$$-\operatorname{Re}\frac{1}{2d}\check{w}^2 = -\frac{1}{2|d|^2}\big(d_1\check{w}_1^2 - d_1\check{w}_2^2 + 2d_2\check{w}_1\check{w}_2\big) = -\frac{1}{2d_1|d|^2}\big((d_1\check{w}_1 + d_2\check{w}_2)^2 - |d|^2\check{w}_2^2\big)$$

by completing the square. It is non-positive if and only if $(d_1\check{w}_1 + d_2\check{w}_2)^2 \geq |d|^2\check{w}_2^2$. In the given situation, we have $\check{w}_2 > 0$ and $d_1\check{w}_1 + d_2\check{w}_2 > 0$, so the condition is $d_1\check{w}_1 + d_2\check{w}_2 \geq |d|\check{w}_2$. This is satisfied since

$$d_2(d_1\check{w}_1 + d_2\check{w}_2) = d_2d_1\check{w}_1 + d_2^2\check{w}_2 \geq d_1^2\check{w}_2 + d_2^2\check{w}_2 = |d|^2\check{w}_2 \geq d_2|d|\check{w}_2$$

(using that $d_2\check{w}_1 \geq d_1\check{w}_2$ and $|d| \geq d_2$).

(ii) This part is shown in the proof of Lemma 3.26.

$\square$

Part (i) of the preceding lemma states that the "interesting region" for $\check{w}$ is not really the whole half-plane described by $d_1\check{w}_2 - d_2\check{w}_1 \leq 0$, but only the set

$$Y_1 := \left\{ \check{w} \in \mathbb{C} \;\middle|\; \check{w}_2 \leq \min\left(\frac{c_2}{c_1}\check{w}_1, \frac{d_2}{d_1}\check{w}_1\right) \right\}.$$

Knowing this, one obtains the following description of the cohomology with compact support of $\mathcal{X}$:

$$\mathrm{H}^l_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq 0 \quad \text{if } l \neq 0$$

and

$$\mathrm{H}^0_{\mathrm{c}}(\mathcal{X};\mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } \check{w} \in Y_1 \text{ and } -\psi_{\mathrm{r}}^+(\check{w}) \leq \check{t} < -\psi_{\mathrm{r}}^-(\check{w}), \\ 0 & \text{otherwise,} \end{cases}$$

with the functions $\psi_{\mathrm{r}}^+, \psi_{\mathrm{r}}^- : \mathbb{C} \to \mathbb{R}$ given by

$$\psi_{\mathrm{r}}^+(w) := \begin{cases} \frac{\check{w}_1^2}{2d_1} & \text{if } \check{w}_1 \leq 0, \\ 0 & \text{if } \check{w}_1 > 0 \end{cases}$$

and

$$\psi_{\mathrm{r}}^-(w) := \begin{cases} \operatorname{Re}\frac{1}{2d}w^2 & \text{if } (c_1d_2 - c_2d_1)\check{w}_2 \leq -(c_1d_1 + c_2d_2)\check{w}_1, \\ \zeta(w) & \text{if } (c_1d_2 - c_2d_1)\check{w}_2 > -(c_1d_1 + c_2d_2)\check{w}_1. \end{cases}$$

It is easily checked that these functions are continuous. Let us check that $\psi_{\mathrm{r}}^+(w) \leq \psi_{\mathrm{r}}^-(w)$

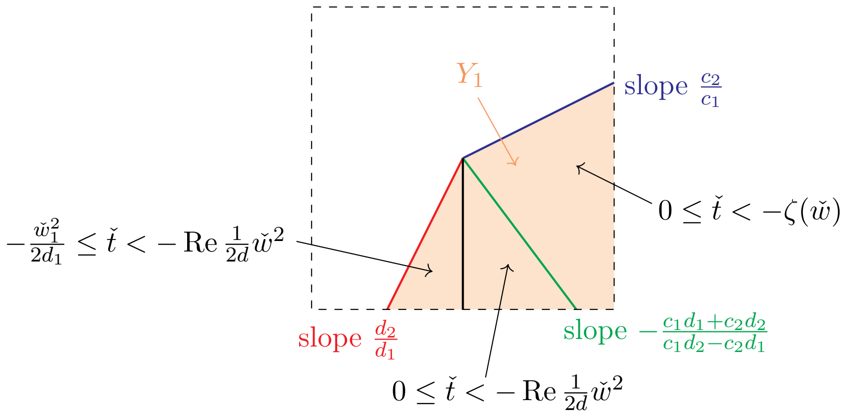


Figure B.5.: An illustration of the regions in the complex plane (with variable $\check{w}$) and the corresponding values of $\check{t}$ for which the cohomology $\mathrm{H}^0_c(\mathcal{X};\mathbf{k})$ is nontrivial. The green line is horizontal in the case $c_1d_2 - c_2d_1 = 0$, and one of the three regions vanishes.

for any $w \in Y_1$. Clearly, $\zeta \le 0$ (cf. the proof of Lemma 3.26 to see that the denominator of ζ is positive), and we also have $\frac{w_1^2}{2d_1} \ge \operatorname{Re}\frac{1}{2d}w^2$ (as in Section B.2.1 with the parameter c). The only thing which is not obvious is therefore the fact that $\operatorname{Re}\frac{1}{2d}w^2 \le 0$ if $w_1 \ge 0$ and $(c_1d_2 - c_2d_1)w_2 \le -(c_1d_1 + c_2d_2)w_1$. We show this as in the proof of Lemma B.3 (i): In the region concerned, both w_2 and $d_1w_1 + d_2w_2 \le 0$ are non-positive (one can easily show that the green line in Fig. B.5 is steeper than the line with slope $-\frac{d_1}{d_2}$). Therefore, we need to show that $d_1w_1 + d_2w_2 \ge |d|w_2$, which works as in the proof above.

B.3.2. Hyperbolae and half-lines

As in Section B.2.2, one is also interested in the spaces given as the intersection of the hyperbolic region (B.3) with the half-lines bounding the sector (B.4).

Horizontal half-line The considerations for the intersection $\mathcal{Y}$ of the hyperbolic region (B.3) with the horizontal half-line

$$x_1 \ge \frac{\check{w}_1}{d_1}, \quad x_2 = \frac{d_1\check{w}_2 - d_2\check{w}_1}{|d|^2}$$

are completely analogous to the one in Section B.2.2 and yield

$$\mathrm{H}^l_c(\mathcal{Y};\mathbf{k}) \simeq 0 \quad \text{if } l \neq 0$$

and

$$\mathrm{H}^0_c(\mathcal{Y};\mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } \check{t} \ge -\psi^+_{\mathrm{r}}(\check{w}), \\ 0 & \text{otherwise.} \end{cases}$$

Non-horizontal half-line The intersection $\mathcal{Y}$ of the hyperbolic region (B.3) with the non-horizontal border of the sector (B.4) given by

$$(c_1 d_2 - c_2 d_1)\left(x_2 - \frac{d_1 \check{w}_2 - d_2 \check{w}_1}{|d|^2}\right) = -c_1 d_1 \left(x_1 - \frac{\check{w}_1}{d_1}\right), \quad x_2 \geq \frac{d_1 \check{w}_2 - d_2 \check{w}_1}{|d|^2}$$

is a space whose cohomology with compact support is described as follows:

As in all the previous cases, noncompact connected components do not contribute to the compactly supported cohomology groups.

If $t < -\operatorname{Re} \frac{1}{2d} \check{w}^2$, the intersection has a compact connected component if and only the half-line's starting point is low enough to intersect the lower branch of the hyperbola.

If $t \geq -\operatorname{Re} \frac{1}{2d} \check{w}^2$, then $\mathcal{Y}$ has a compact connected component if and only if the half-line intersects the hyperbola's right branch in two distinct points (see Section B.3.1).

The result (also similar to Section B.2.2) is:

$$\mathrm{H}^l_{\mathrm{c}}(\mathcal{Y}; \mathbf{k}) \simeq 0 \quad \text{if } l \neq 0$$

and

$$\mathrm{H}^0_{\mathrm{c}}(\mathcal{Y}; \mathbf{k}) \simeq \begin{cases} \mathbf{k} & \text{if } c_1 \check{w}_2 - c_2 \check{w}_1 \leq 0 \text{ and } 0 \leq \check{t} < -\zeta(\check{w}), \\ 0 & \text{otherwise.} \end{cases}$$

List of notations

$\mathbb{1}$	an identity matrix.
$\arg z$	the argument of a complex number $z \neq 0$.
$\mathbb{C}$	the complex plane equipped with the Euclidean topology.
$\mathbb{C}^\times$	the set of complex units, $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$.
C	the set of parameters for a D-module of pure Gaussian type, a finite (and nonempty) subset of $\mathbb{C}^\times$.
can	a canonical morphism.
$\mathcal{D}_X$	the sheaf of linear partial differential operators with holomorphic coefficients on X.
$\mathrm{D}^\mathrm{b}(\mathcal{D}_X)$	the derived category of left $\mathcal{D}_X$-modules, see Section 1.3.
$\mathrm{D}^\mathrm{b}_\mathrm{hol}(\mathcal{D}_X)$	the full subcategory of $\mathrm{D}^\mathrm{b}(\mathcal{D}_X)$ of objects with holonomic cohomologies.
$\mathrm{D}^\mathrm{b}(\mathbf{k}_X)$	the derived category of sheaves of $\mathbf{k}$-vector spaces on X.
$\mathrm{D}^\mathrm{b}(\mathrm{I}\mathbf{k}_X)$	the derived category of ind-sheaves on X.
$\mathrm{E}^\mathrm{b}(\mathrm{I}\mathbf{k}_X)$	the category of enhanced ind-sheaves on X, see Section 1.2.1.
$\mathcal{E}^\varphi_{U\|X}$	an exponential D-module, see Definition 1.6.
$\mathsf{E}^\varphi_{Z\|X}, \mathsf{E}^{\varphi^+ \triangleright \varphi^-}_{Z\|X}$	exponential enhanced sheaves, see Definition 1.3.
$\mathbb{E}^\varphi_{Z\|X}, \mathbb{E}^{\varphi^+ \triangleright \varphi^-}_{Z\|X}$	exponential enhanced ind-sheaves, see Definition 1.3.
$\tilde{i}$	the inclusion $\tilde{i}\colon \mathbb{C} \times \mathbb{R} \hookrightarrow \mathbb{P} \times \mathbb{R}$.
ι_X	the natural embedding from sheaves to ind-sheaves $\iota_X\colon \mathrm{D}^\mathrm{b}(\mathbf{k}_X) \to \mathrm{D}^\mathrm{b}(\mathrm{I}\mathbf{k}_X)$.

$\mathbf{k}$	the field of complex numbers, $\mathbf{k} = \mathbb{C}$.
$\mathbf{k}_X$	the constant sheaf with stalk $\mathbf{k}$ on X.
$\mathbf{k}_X^{\mathrm{E}}$	the object $``\varinjlim_{a\to\infty}" \, \mathbf{k}_{\{t\geq a\}} \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$.
${}^{\mathsf{L}}(\bullet)$, ${}^{\mathcal{L}}(\bullet)$	the functors of Fourier–Laplace and enhanced Fourier–Sato transform, see Section 3.1.
$\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$	the category of holonomic left $\mathcal{D}_X$-modules.
$\mathbb{P}$	the complex projective line equipped with the analytic topology (= the Riemann sphere).
π	the projection $X \times \mathbb{R} \to X$; or the real number π, representing an angle of $180°$.
$\overline{\mathbb{R}}$	the two-point compactification of the real line, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
r_c	the rank of the regular part R_c associated to $c \in C$ in the Levelt–Turrittin decomposition of a D-module of pure Gaussian type C.
r	the sum of the ranks r_c, i.e. $r = \sum_{c\in C} r_c$.
$\mathbb{r}$	the family of the ranks r_c, i.e. $\mathbb{r} = (r_c)_{c\in C}$.
$\mathrm{SingSupp}(\mathcal{M})$	the singular support of the D-module $\mathcal{M}$.
$\mathcal{S}ol_X^{\mathrm{E}}$	the enhanced solution functor, see Theorem 1.10.
$\overset{+}{\otimes}$, $\overset{*}{\otimes}$	the convolution product for enhanced ind-sheaves and enhanced sheaves.
$\subset\subset$	"is a relatively compact subset of".
$<_S$, $<_\theta$	a certain order relation on the parameter set C, see Notation 2.6.
$[a,b] \subseteq \mathbb{R}/2\pi\mathbb{Z}$	the image of the interval $[a,b] \subseteq \mathbb{R}$ under the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$.
(f,g)	the morphism $A \to B \oplus C$ given by $f\colon A \to B$ and $g\colon A \to C$ through the universal property of the product.
$f \pm g$	the morphism $A \oplus B \to C$ given by $f\colon A \to C$ and $\pm g\colon B \to C$ through the universal property of the coproduct.
$f\vert g$, $f\!\downarrow\! g$	the morphism $A \oplus B \to C \oplus D$ given as the direct sum of $f\colon A \to C$ and $g\colon B \to D$.

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